A derived graph: Triangle graph of a graph

by

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

______________________________ (Signature of candidate)

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I would like to thank my supervisor Eunice Mphako-Banda for introducing me to the world of graph theory and especially for all the effort spent to get me to understand the details involved in producing an extended mathematical argument.

— Elias Sithole

Abstract

In this dissertation we study a derived graph called the triangle graph of a graph. We start by giving a brief background on derived graphs, followed by some basic definitions in graph theory which are relevant in this work. We then discuss some well known classes of graphs and some well known graph operations.

Thereafter, we give a known formal definition of a line graph, followed by a few examples and some well known results on line graphs. This leads to the introduction of the main structure of this work, the triangle graph of a graph. We define the triangle graph of a graph and clarify the concept with examples. Then we establish a few properties of a triangle graph of a graph, followed by establishing triangle graphs of certain classes of graphs.

Finally, we conclude the dissertation by discussing vertex-join of a graph and the relationship between the graph G, and the triangle graph of a vertex-join of G.

— Elias Sithole

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Chapter 1

Introduction

In this chapter, we give a brief background on derived graphs. Thereafter, we give an overview of the dissertation, followed by some basic definitions in graph theory which are relevant to this work. Furthermore, we list some well known classes of graphs. We conclude this chapter by stating some known graph operations.

1.1 Background on derived graphs

> The whole universe is based on the concept of graph theory where love is an edge, that is connecting two vertices or people either directly or indirectly.
>
> Yatin Mehndiratta

It is no coincidence that there is a vast interest in graph theory. Its wide range of applicabilites makes it one of the most intriguing concepts in modern day applied mathematics. The quotation by Yatin Mehndiratta, for instance, shows graph theory being applied in the field of psychology. Although this background is not about the general history of graph theory, it is however necessary to mention Leonhard Euler. Many authors regard Leonhard Euler as the father of graph theory. The Königsberg bridge problem which can be traced back as early as the year 1736 was introduced by Leonhard Euler. Within graph theory, there are various interesting topics worth studying, one particular topic which is interesting and relevant to our study is derived graphs. There is a wide range of problems which can be studied on derived graphs, for instance studying structural properties of the derived graph in comparison to the original graphs. We now give a brief history of derived graphs. Not only will this background highlight some important historical events, but it will also lay out a foundation of the subject matter of this dissertation. It is important to note that a derived graph is obtained from a certain graph operation. It is from different kinds of graphs operations that we are able to derive graphs.

We first discuss graph operations of which they were independently discovered by different authors. We begin with two of the most common graph operations, the graph union and graph join. Not much is known in terms of the history of these two operations, however for more information, see [9]. Another graph operation which is widely known is graph product, but comes in many different forms. There is a handbook on products of graphs, hence for definitions, examples and most theory on products, we refer to [8]. There are four fundamental graph products, namely, cartesian product, direct product, lexicographic product and strong product. All of which each were introduced by a different author, except for the cartesian product and direct product.

Introduced in 1912 by A.N. Whitehead and B. Russell, the cartesian product has been widely studied and regarded as the simplest graph product. The direct product also referred to as the tensor product or kronecker product, was introduced by A.N. Whitehead and B. Russell and traces back to the same year as the cartesian product, 1912. It is important to know that different authors use different notations for these products. Another well known graph product is the lexicographic product, which dates back to the year 1914. Finally, the strong product was introduced in 1960 by G. Sabidussi. It was only around the years 1960 - 1961 that authors began studying graph products actively. Some very interesting problems in graph theory arise from just studying these products alone. In 1960, G. Sabidussi came up with a very intriguing theory. One of the questions which was addressed is, do graphs factor uniquely into primes over a given product? G. Sabidussi and V.G. Vizing went on to show that a connected graph which is a cartesian product can be uniquely factored into primes, which is now called the *Sabidussi-Vizing theorem*. In 1986, J. Feigenbaum and A. Schäffer, showed that prime factorization is not easy for the lexicographic product and went further in 1992 to show that it is possible for graphs to factor uniquely into primes over the strong product. W. Imrich, used the very same approach in 1998 for the direct product.

Another graph operation known as line graph is one of the most well known graph operations. Line graphs were first introduced by H. Whitney [20], traced back to the year 1932. Although H. Whitney was the first person to introduce line graphs, H. Whitney never coined this name. We will give more details on the history of line graphs in Chapter 2.

Another well known graph operation is the intersection of graphs. Just like with graph products, there are different classes of intersection graphs. A class of intersection graphs called interval graphs was discovered by S. Benzer, see [4]. He saw it fitting that a string of genes representing a bacterial chromosome be regarded as a closed interval on the real line. Interestingly, this led to another discovery of a class of intersection graphs called string graphs. In 1959, G. Hajós [9], proposed that a graph can be associated with every finite family F of intervals. C. Lekkeikerker and J. Boland [12] continued with this work in 1962 and so did P.C. Gilmore and A.J. Hoffman [7] in 1964. Another class of intersection graphs called circular arc graphs, appeared in the year 1964 in the paper published by H. Hadwiger [11]. More publications followed thereafter, most notably by W.E. Klee in 1969 and A. Tucker who wrote on circular graphs in a Ph.D thesis in 1969.

We now mention chordal graphs which are also classified as intersection graphs, first studied by A. Hajnal and J. Suranyl [1] in 1958. Back then chordal graphs were mostly referred to as rigid circular graphs or triangulated graphs. In 1960, C. Berge [1] went on to introduce the notion of perfect graphs which also involves chordal graphs. There are many other classes of intersection graphs which we will not go through their backgrounds, for example, circle graphs, trapezoid graphs, disk graphs, tolerance graphs, permutation graphs, etc.

In this dissertation, we derive our own graph called a triangle graph of a graph. The dissertation, will be centered around some results in line graphs which will be mimicked for a triangle graph of a graph.

1.2 Overview

Chapter 1 is a brief introduction to graph theory. We introduce the main structure of this work, a graph. We then give some basic definitions in graph theory along with examples illustrating these definitions. In Section 1.4, we study classes of graphs which are relevant in this dissertation. Finally, in Section 1.5, we give some commonly known graph operations, in particular we discuss the concept of intersection graphs.

In Chapter 2, we start by giving a brief history on line graphs. Then, we describe and define line graphs followed by a few examples. We discuss line graphs of some classes of graphs in Section 2.4, and conclude the section by stating and proving some properties of line graphs of classes of graphs.

Our main objects of study appear in Chapter 3, where we start by establishing notations for the rest of the dissertation. We then give a formal definition of a triangle graph of a graph followed by examples and some properties. In all the other sections that follow, we study triangle graphs of some classes of graphs. We state and prove some properties of these classes of graphs.

Chapter 4, investigates the triangle graph of a vertex-join. We begin by giving a formal definition of a vertex-join of a graph. We then state and prove some properties of the triangle graph of a vertex-join followed by some examples of classes.

In Chapter 5, we conclude this dissertation and point out some problems that emerged from this study which may be of particular interest for further study.

1.3 Basic definitions

In this section, we define some concepts in graph theory which are of particular interest in this dissertation. For the definitions in this section, we refer the reader to [3, 6, 9], unless otherwise stated.

Definition 1.3.1. A *graph* is an ordered pair $G = (V, E)$, where V is a non-empty set of elements called vertices of G and E. The elements of E are called edges of G.

In graph theory, we use the idea of representing vertices as points and edges as lines to form a graph. We often write $V(G)$ to denote the *vertex set* of G and $E(G)$, to denote the *edge set* of G. The edge $\{p, q\}$, will join vertices p and q in G. Alternatively, we denote the edge $\{p, q\}$ by pq or qp. We say vertices p and q are *endpoints* of the edge pq. Two vertices p and q are said to be *adjacent* to each other if they have a common edge. Two distinct edges are adjacent or incident if they have a common end vertex. If we have $e = pq$ then e is said to be *incident* to vertices p and q.
The number of vertices in a graph is called the *order* of the graph and is denoted by $|V(G)|$. The number of edges in a graph is called the *size* of the graph and is denoted by $|E(G)|$.

Definition 1.3.2. A set of two or more edges connected to the same pair of vertices is called *multiple* or *parallel* edges. An edge with one repeated endpoints is called a *loop*. A *simple* graph is a graph with no loops and multiple edges. Two adjacent vertices are called *neighbors*. The set of all neighbors of a vertex p is called the *open neighborhood* denoted by $N(p)$. The *closed neighborhood* of a vertex p is given by $N[p] = N(p) \cup \{p\}$.

Definition 1.3.3. The *degree* of a vertex p, is the number of all the edges incident to the vertex p and is denoted by $deg(p)$. If a graph G has a loop at vertex p then when counting the degree of the vertex p, the loop will be counted twice. The largest degree of a graph G is called the *maximum degree* denoted $\Delta(G)$ and the *minimum degree* is denoted by $\delta(G)$. A *degree sequence* of a graph G, is simply a list of all degrees of each vertex in $V(G)$. The sequence can be given in either nonincreasing or nondecreasing order.

Definition 1.3.4. A graph G is a *subgraph* of a graph F if $V(G) \subseteq V(F)$ and $E(G) \subseteq E(F)$. If G is a subgraph of F we can write $G \subseteq F$. G is a *spanning subgraph* of F if $V(G) = V(F)$. G is an *induced subgraph* of F if every edge of F with endpoints in $V(G)$ is also an edge in G, hence we denote it by $G[F]$.

Definition 1.3.5. A graph G is *bipartite* if $V(G)$ can be partitioned into two sets, X and Y such that every edge of G is incident to a vertex of the set X and a vertex of set Y.

Example 1.3.6. Consider the graph G given in Figure 1.1. The vertex set of the graph G is $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ and the edge set $E(G) = \{e_1, v_1v_2, v_2v_3, v_2v_5, v_3v_4, v_4v_7, v_5v_6, v_6v_7, v_7v_8, e_{10}\}$. The order of G is $|V(G)| = 8$ and the size is $|E(G)| = 10$. Edges e_1 and e_{10} are loops. The degrees of the vertices are as follows: $deg(v_1) = deg(v_2) = deg(v_7) = deg(v_8) = 3$ and $deg(v_3) = deg(v_4) = deg(v_5) = deg(v_6) = 2$. Hence the degree sequence is $3, 3, 3, 3, 2, 2, 2, 2$. The open neighborhood of vertex v_7 is given by $N(v_7) = \{v_4, v_6, v_8\}$, while the closed neighborhood is given by, $N[v_7] = \{v_4, v_6, v_8\} \cup \{v_7\}$.

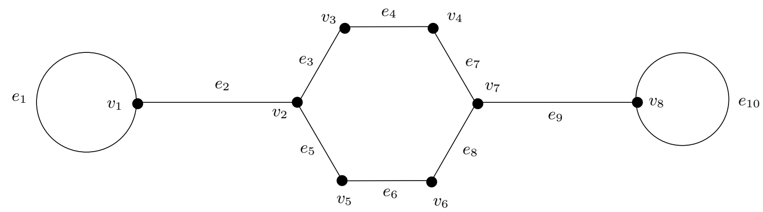


Figure 1.1: A graph G.

Definition 1.3.7. A *walk* is a way of getting from one vertex to another, consisting of an alternating sequence of vertices and edges, $v_0e_1v_1e_2v_2\dots e_nv_n$, such that all consecutive vertices are adjacent to each other. If all the edges of a walk are different, then we refer to it as a *trail*. If all the vertices of a trail are different, then we refer to it as a *path*. A walk is *closed* if the first and last vertices are the same. If the first and last vertices are distinct, then we refer to it as an *open* walk. A *cycle* is a trail with all vertices different, however the first and last vertices are not different.

Example 1.3.8. Consider the graph G given in Figure 1.2. Sequence $v_1bv_2fv_6fv_2hv_3$ is a walk which is not a trail. Sequence $v_1bv_2fv_6gv_5ev_2bv_3$ is a trail however it is not a path. Sequence $v_1bv_2fv_6gv_5$ is a path as we can see that the vertices and the edges are all different. Sequence $v_2hv_3iv_5cv_4dv_2$ is a cycle since the first and last vertices are the same.

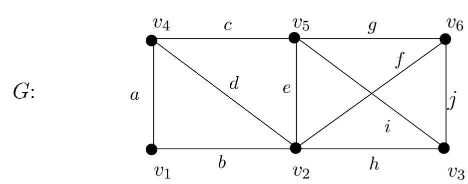


Figure 1.2: A graph to illustrate walks.

Definition 1.3.9. A graph is said to be a *connected* graph if there is a path from any one vertex to any other vertex. A graph that is not connected is a *disconnected* graph.

Example 1.3.10. The graphs in Figure 1.3 are examples of a connected and a disconnected graph.

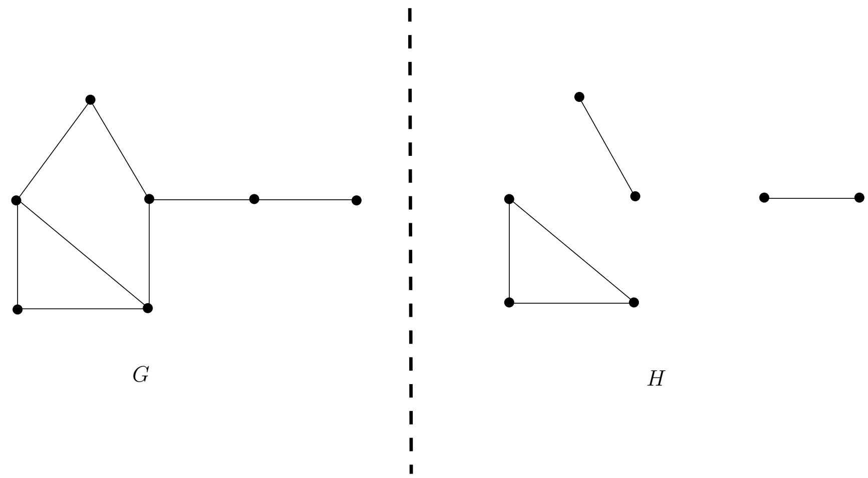


Figure 1.3: A connected graph G and disconnected graph H.

Definition 1.3.11. A *bridge* is an edge in a graph G that when deleted, the resulting graph is disconnected.

Definition 1.3.12. Consider the function Θ with the mapping $\Theta : V(G_1) \to V(G_2)$ where G_1 and G_2 are two simple graphs. G_1 is *isomorphic* to G_2 if Θ is a bijection which preserves adjacency and non-adjacency. Thus, we have $uv \in E(G_1) \Leftrightarrow \Theta(u)\Theta(v) \in E(G_2)$ and we write $G_1 \cong G_2$.

1.4 Classes of graphs

In this section, we discuss some well known classes of graphs. For the definitions in this section, we refer the reader to [3, 6, 9, 16], unless otherwise stated.
A class of graphs is a set of graphs having certain specified properties. We discuss, a few classes of graphs which are basic and relevant to this dissertation. We begin with a class of graphs known as trees.

Definition 1.4.1. We define a *tree* as a graph in which any two vertices are connected by a path and has no cycles. We denote a tree on n vertices as t_n. A *forest*, is defined as a union of trees. Some trees are classified based on their properties. A graph of order n with vertex set $V = \{v_1, v_2, \ldots, v_{n-1}, x\}$ and edge set $E = \{xv_1, xv_2, \ldots, xv_{n-1}\}$ is a tree called a *star*, denoted by S_n. A *path* is a tree of order n, with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{v_iv_{i+1} \mid \text{for } \ i \in \{1, 2, \ldots, n-1\}\ \}$, denoted by P_n.

It is important to note that a tree has no multiple edges and loops, thus a tree is a simple graph.

Example 1.4.2. The graphs given in Figure 1.4 are examples of trees.

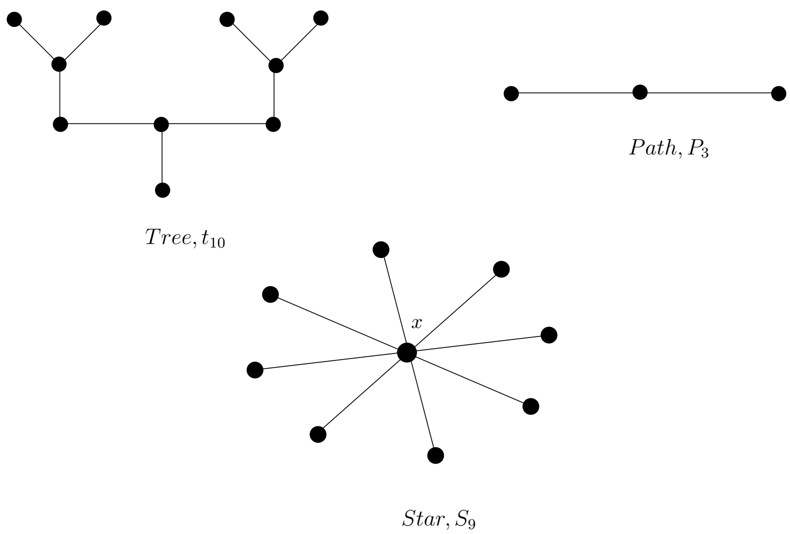


Figure 1.4: Trees.

Definition 1.4.3. A *null* graph of order n is a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{\} = \emptyset$, denoted by N_n.

Example 1.4.4. The graphs given in Figure 1.5 are examples of null graphs.

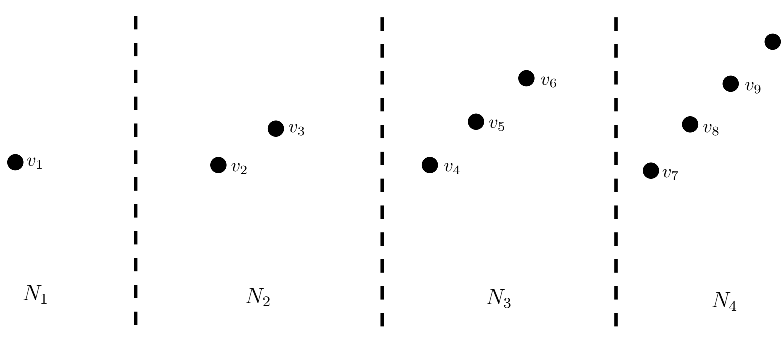


Figure 1.5: Null graphs.

Definition 1.4.5. A *complete* graph is a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{v_i v_j \mid i \neq j \ i, j \in \{1, 2, \ldots, n\}\}$, denoted by K_n.

Note that the degree of each vertex of a complete graph K_n is $n-1$.

Example 1.4.6. The graphs given in Figure 1.6 are examples of complete graphs namely; K_1, K_2, K_3, and K_4.

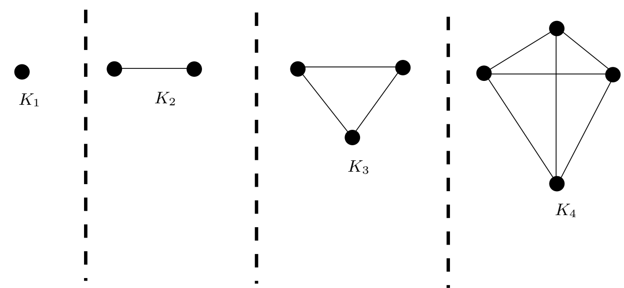


Figure 1.6: Complete graphs.

Definition 1.4.7. A *cycle* graph of order n is a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{v_1v_2, v_2v_3, v_3v_4, \ldots, v_{n-1}v_n, v_nv_1\}$, denoted by C_n. Note that a cycle graph is isomorphic to a closed trail with all the vertices different except for the first and last vertex. In particular, C_3 is often referred to as a triangle, C_4 referred to a square and C_5 referred to a pentagon.

Example 1.4.8. The graphs given in Figure 1.7 are examples of cycle graphs namely; C_3, C_4 and C_5.

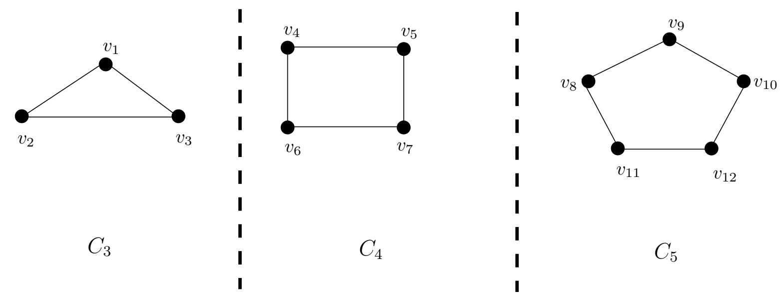


Figure 1.7: Cycle graphs.

Definition 1.4.9. A graph G is called *regular* if all the vertices have the same degree, thus a graph is k-regular if every vertex of G has degree k.

Note that a complete graph of order n, is an $(n-1)$-regular graph and any cycle graph is 2-regular.

Example 1.4.10. The graphs given in Figure 1.8 are examples of regular graphs namely; 0-regular, 1-regular, 2-regular and 3-regular.

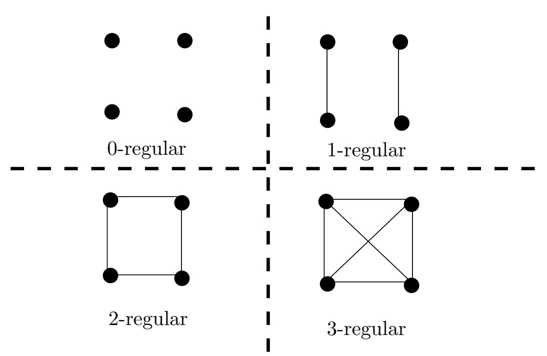


Figure 1.8: Regular graphs.

Definition 1.4.11. A *wheel* graph of order n is a graph with vertex set $V = \{v_1, v_2, \ldots, v_n, x\}$ and edge set

$$E = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_1, x\}, \{v_2, x\}, \ldots, \{v_{n-1}, x\}\},$$

denoted by W_n. Note that the edge of the form v_ix is called a spoke, while the edge of the form v_iv_{i+1} where $i \in \{1, 2, \ldots, n, n-1\}$ is called an arc of a wheel graph. For further details on wheel graphs, we refer the reader to [16].

Example 1.4.12. The graphs given in Figure 1.9 are examples of wheel graphs namely, W_4 and W_5.

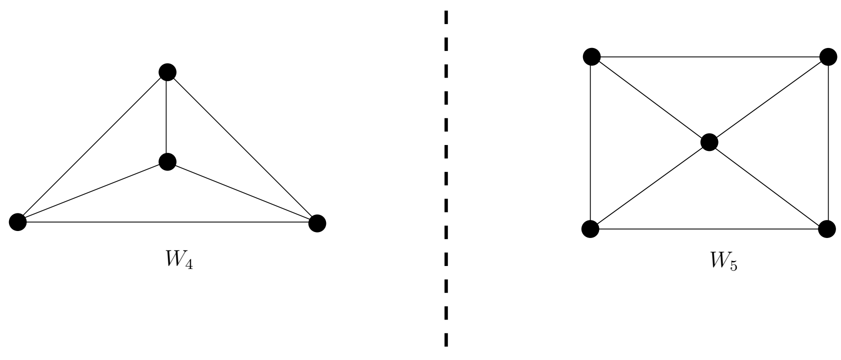


Figure 1.9: Wheel graphs W_4 and W_5.

Definition 1.4.13. A *flower* graph is a graph with vertex set $V = \{1, 2, \ldots, n, n+1, \ldots, n(m-1)\}$ and edge set

$$E = \{\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}\} \cup \{\{1,n+1\},\{n+1,n+2\},\{n+2,n+3\},$$
$$\{n+3,n+4\},\ldots,\{n+m-3,n+m-2\},\{n+m-2,2\}\} \cup \{\{2,n+m-1\},\{n+m-1,n+m\},$$
$$\{n+m,n+m+1\},\{n+m+1,n+m+2\},\ldots,\{n+2(m-2)-1,n+2(m-2)\},$$
$$\{n+2(m-2),3\}\} \cup, \ldots, \cup \{\{n,n+(n-1)(m-2)+1\},$$
$$\{n+(n-1)(m-2)+1,n+(n-1)(m-2)+2\},\{n+(n-1)(m-2)+2,n+(n-1)(m-2)+3\},$$
$$\{n+(n-1)(m-2)+3,n+(n-1)(m-2)+4\},\ldots,\{nm-1,nm\},\{nm,1\}\}.$$

Note that the m-cycles are called the petals and the n-cycle is called the center of $f_{n\times m}$.

The order of $f_{n\times m}$ is given by $|V(G)| = n(m-1)$ and the size by $|E(G)| = nm$. For further details on flower graphs, we refer the reader to [16].

Example 1.4.14. The graphs given in Figure 1.10 are examples of flower graphs namely; $f_{4\times 4}$ and $f_{4\times 6}$.

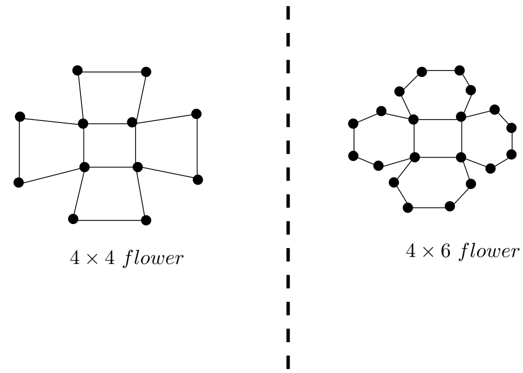


Figure 1.10: Flower graphs.

Definition 1.4.15. A *Plane* graph is a graph which is given on a $\mathbb{R}^2$ plane and where none of the edges of the graph cross each other.

Definition 1.4.16. A *Planar* graph is a graph that is isomorphic to a plane graph.

For further details on plane and planar graphs, we refer the reader to [3].

Example 1.4.17. The two graphs given in Figure 1.11 are examples of a planar and a non planar graph.

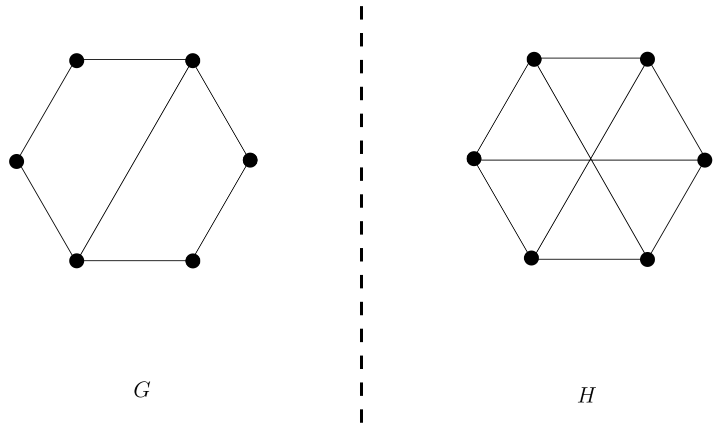


Figure 1.11: A planar graph G and non planar graph H.

1.5 Graph operations

In this section, we discuss some of the most commonly used graph operations in graph theory which are relevant to this work. For the definitions in this section, we refer the reader to [2, 3, 8, 9].

1. The *union* of graph G_1 and graph G_2, denoted by $G_1 \cup G_2$ is a graph G with vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2)$.

   Example 1.5.1. The graph in Figure 1.12 is an example of a union graph $G = G_1 \cup G_2$, with vertex set $V(G) = \{v_1, v_2, v_3, v_4, v_5, u_1, u_2, u_3, u_4\}$ and edge set $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\} \cup \{h_1, h_2, h_3, h_4, h_5\}$.

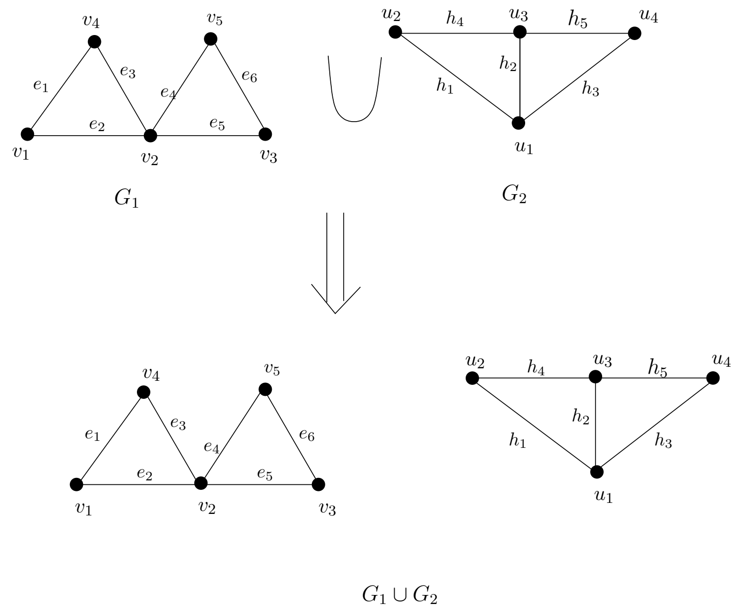


Figure 1.12: The union $G_1 \cup G_2$.

2. The *intersection* of graph G_1 and graph G_2, denoted by $G_1 \cap G_2$ is a graph G with vertex set $V(G) = V(G_1) \cap V(G_2)$ and edge set $E(G) = E(G_1) \cap E(G_2)$.

   Example 1.5.2. The graph in Figure 1.13 is an example of the intersection of graphs $G = G_1 \cap G_2$, with vertex set $V(G) = \{a, b\}$ and $E(G) = ab$.

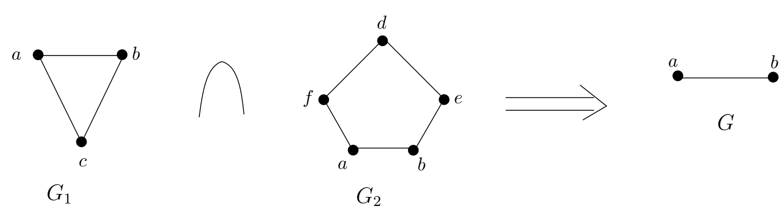


Figure 1.13: The intersection $G_1 \cap G_2$.

3. The *cartesian product* of graph G_1 and graph G_2, denoted by $G_1 \times G_2$ is a graph G with vertex set $V(G) = V(G_1) \times V(G_2) = \{(u_i, v_j) | u_i \in V(G_1), v_j \in V(G_2)\}$

and the edge set $E(G)$ has elements $\{(u_i, v_j), (u_t, v_s)\}$ where $u_i = u_t$ and $v_j v_s$ is an edge in G_2 or $v_j = v_s$ and $u_i u_t$ is an edge in G_1.

4. *The lexicographic product* of two graphs G_1 and G_2, denoted by $G \circ H$ is obtained from the cartesian product by adding extra edges of the form, vertices (u_i, v_i) and (u_j, v_j) are adjacent if $u_i u_j \in E(G_1)$ and $v_i v_j \in E(G_2)$.

Example 1.5.3. Consider the graph G_1 with vertex set $V(G_1) = \{a, b\}$ and graph G_2 with vertex set $V(G_2) = \{0, 1, 2\}$ in Figure 1.14. The graph $G = G_1 \times G_2$ in Figure 1.15 is an example of the cartesian product of graph G_1 and graph G_2, with vertex set $V(G) = \{(a, 0), (a, 1), (a, 2), (b, 0), (b, 1), (b, 2)\}$ and edge set $E(G) = \{00, 01, 01, 12, 12, 22\}$. The graph $G_1 \circ G_2$ in Figure 1.15 is an example of the lexicographic product of graph G_1 and graph G_2 with vertex set $V(G_1 \circ G_2) = \{(a, 0), (a, 1), (a, 2), (b, 0), (b, 1), (b, 2)\}$ and edge set $E(G_1 \circ G_2) = \{00, 01, 01, 12, 12, 22, a1, a2, a2, b1, b1, b2\}$.

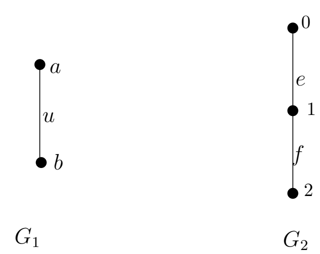


Figure 1.14: Graphs G_1 and G_2.

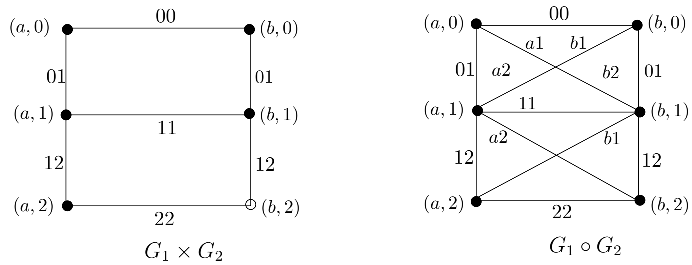


Figure 1.15: Graphs $G_1 \times G_2$ and $G_1 \circ G_2$.

5. The *ring sum* of G_1 and G_2, denoted by $G_1 \oplus G_2$, has vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2) - E(G_1) \cap E(G_2)$. Thus for the edge set of the ring sum we take edges that are either in G_1 or G_2 but not in both.

   Example 1.5.4. Consider the graphs in Figure 1.16. $V(G_1) = \{v_1, v_2, v_3, v_4, v_5\}$, $E(G_1) = \{a, b, c, d, e, f, g\}$ and $V(G_2) = \{v_1, v_3, v_4, v_6\}$, $E(G_2) = \{f, g, h, i, j, k\}$. The graph $G_1 \oplus G_2$ is an example of the ring sum of graph G_1 and graph G_2, with vertex set $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set $E(G) = \{a, b, c, d, e, h, i, j, k\}$.

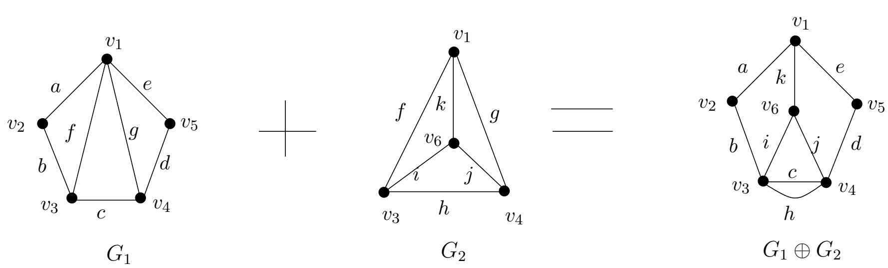


Figure 1.16: The graph $G_1 \oplus G_2$.

6. The *complement* of the graph G, denoted by $\overline{G}$ is a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{v_i v_j \mid v_i v_j \notin E(G)\}$.

   Example 1.5.5. Consider the graphs in Figure 1.17, graph G has vertex set

$V = \{a, b, c, d, e, f\}$ and edge set $E = \{ac, ae, bd, bf, ce, df\}$, while the complement of G has vertex set $V = \{a, b, c, d, e, f\}$ and edge set

$$E = \{ab, ad, af, bc, be, cd, cf, de, ef\}.$$

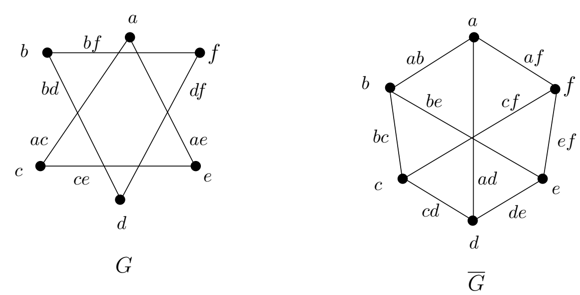


Figure 1.17: A graph G and the complement $\overline{G}$.

7. A *chord* of a cycle of four or more vertices in a graph, is an edge connecting two vertices of the cycle which are not adjacent in the cycle. A *chordal* graph is a graph where every cycle of length four or more has a cycle chord. Note that a chordal graph is not unique.

Example 1.5.6. The graphs in Figure 2.1 are examples of chordal graphs.

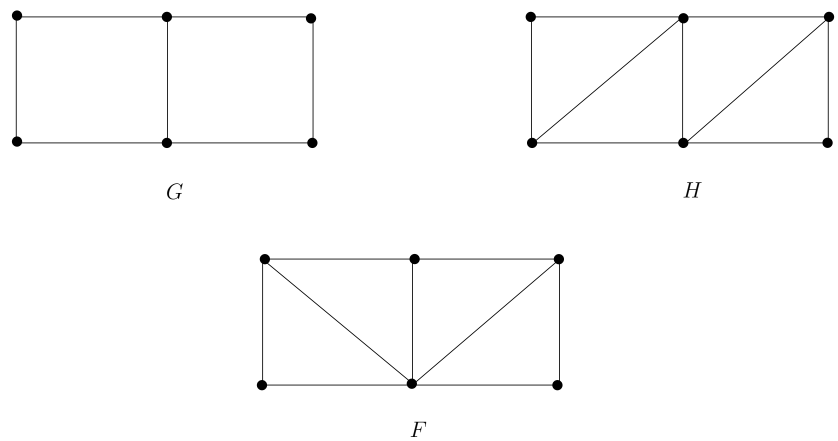


Figure 1.18: Graph G and the chordal graph, H and F of G.

1.6 Intersection graph

In this section, we introduce another form of creating graphs, called intersection graphs. For the definitions in this section, we refer the reader to [9, 14].

Definition 1.6.1. Let $F = \{f_1, f_2, \ldots, f_n\}$ be a set of sets. An *intersection* graph G is a graph with the vertex set $V(G) = \{f_1, f_2, f_3, \ldots, f_n\}$ and edge set $E(G) = \{f_1 f_2 \mid f_1 \cap f_2 \neq \emptyset\}$.

There are many classes of intersection graphs in the literature. In this section, we give one example of intersection graph called interval graphs.

Definition 1.6.2. A graph G is said to be an *interval* graph, if a set of intervals on a real line can be put into one-to-one correspondence with a set of vertices. Two vertices will be adjacent to each other if the intervals have non-empty intersection.

Example 1.6.3. The graph in Figure 1.19 is an example of an interval graph. Each interval corresponds to a vertex in graph G.

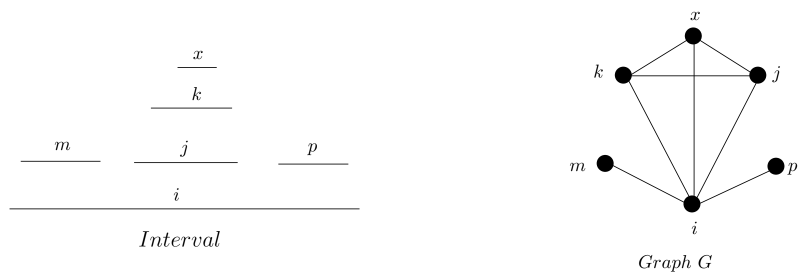


Figure 1.19: Interval graph.

Chapter 2

Line graphs

2.1 Introduction

In this chapter, we discuss the concept of a line graph of a graph. We begin by giving a brief background of line graphs. Then, we give a formal definition of a line graph, illustrating with examples. Thereafter, we give some properties of line graphs followed by line graphs of some known classes of graphs.

2.2 Background on line graphs

It is not clearly known as to when exactly, line graphs were introduced, however, we trace it back as early as the year 1932, see [20]. We mentioned in Section 1.1, that H. Whitney was the first person to introduce the concept of line graphs. In 1932, H. Whitney proved that a connected graph can be completely removed from its line graph under a special case, see [20]. Different authors followed on H. Whitney's work in which most rediscovered the concept and many came up with their own names, Ø. Ore used the name *interchange graphs*, G. Sabidussi used the name *derivative graphs* and L.W. Beineke used the name *derived graphs* just to mention a few, see [9]. The name line graph was first used by F. Harary and R.Z. Norman in 1960, [10], but before that line graphs were just commonly known as edge graphs.

2.3 Line graph of a graph

In this section, we give a formal definition of a line graph, followed by illustrations of this concept through examples. We then give some well known properties of line graphs. For further details of the discussion in this section, we refer the reader to [3, 9, 19] unless stated otherwise.

Definition 2.3.1. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{v_i v_j = e_{ij} \mid \text{ for some } i, j \in \{1, \ldots, n\}\}$. The line graph of the graph G, denoted by $L(G)$ is a graph with vertex set $V(L(G)) = E(G)$ and edge set $E(L(G)) = \{e_{ij}e_{ik} \mid e_{ij},\ e_{ik}\ \in E(G)\}$.

It is important to note that edge $e_{ij} = e_{ji}$ since v_i and v_j are end points of an edge without any order. Hence we can rewrite the edge set of the line graph as $E(L(G)) = \{e_{ij}e_{lk} \mid e_{ij},\ e_{lk}\ \in E(G)\}$ where $i \in \{l, k\}$ or $j \in \{l, k\}$.

Example 2.3.2. The graph G_1 given in Figure 2.1 has vertex set $V(G_1) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and using the notation in Definition 2.3.1, has edge set $E(G_1) = \{e_{12}, e_{23}, e_{24}, e_{34}, e_{35}, e_{45}, e_{56}\}$.

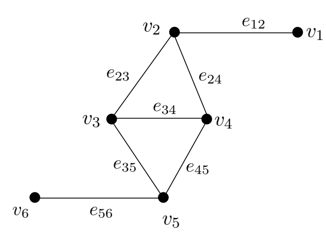


Figure 2.1: Graph G_1.

We now construct the line graph $L(G_1)$ of the graph G_1 given in Figure 2.2. $L(G_1)$ has vertex set $V(L(G_1)) = E(G_1) = \{e_{12}, e_{23}, e_{24}, e_{34}, e_{35}, e_{45}, e_{56}\}$ and has edge set

$E(L(G_1)) = \{e_{12}e_{23}, e_{12}e_{24}, e_{23}e_{24}, e_{23}e_{35}, e_{23}e_{34}, e_{24}e_{34}, e_{24}e_{45}, e_{34}e_{45}, e_{34}e_{35}, e_{35}e_{45}\}$
$\cup \{e_{35}e_{56}, e_{45}e_{56}\}$.

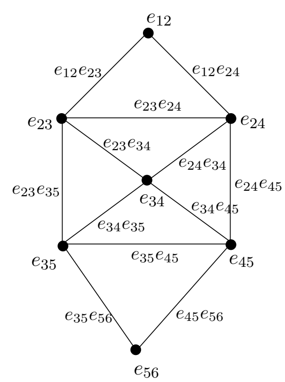


Figure 2.2: The line graph $L(G_1)$ of graph G_1.

Example 2.3.3. The graph G_2 given in Figure 2.3 has vertex set $V(G_2) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and using the notation in Definition 2.3.1, has edge set $E(G_2) = \{e_{13}, e_{23}, e_{34}, e_{35}, e_{56}, e_{57}, e_{67}\}$.

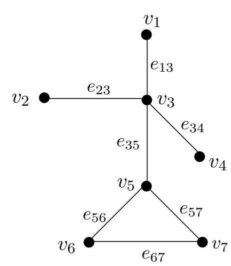


Figure 2.3: Graph G_2.

We now construct the line graph $L(G_2)$ of the graph G_2 given in Figure 2.4. $L(G_2)$ has vertex set $V(L(G_2)) = \{e_{13}, e_{23}, e_{34}, e_{35}, e_{56}, e_{57}, e_{67}\}$ and edge set
$E(L(G_2)) = \{e_{13}e_{23}, e_{13}e_{34}, e_{13}e_{35}, e_{23}e_{34}, e_{23}e_{35}, e_{34}e_{35}, e_{35}e_{56}, e_{35}e_{57}, e_{56}e_{57}\} \cup$
$\{e_{56}e_{67}, e_{57}e_{67}\}$.

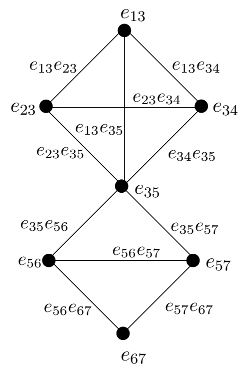


Figure 2.4: The line graph $L(G_2)$ of graph G_2.

The following proposition, states without proof, some well known basic properties of line graphs.

Proposition 2.3.4. *Let G be a graph and $L(G)$ the line graph of G. Then*

(a) G is connected if and only if $L(G)$ is connected.

(b) $L(H)$ is a subgraph of $L(G)$ if H is a subgraph of G.

(c) The degree of vertex e in $L(G)$ is given by $d_G(u) + d_G(v) - 2$, where $e = uv$ is an edge of the graph G.

We state the following well known lemma which will be used in proving Proposition 2.3.7.

Lemma 2.3.5. *Let G be a graph of order n and size m with degree sequence $d_1, d_2, \ldots, d_n$. Then*

$$m = \frac{1}{2}\sum_{i=1}^{i=n} d_i.$$

Proof. Consider an edge $e = uv$ of graph G. Let $d_1 = d_G(u)$ and $d_2 = d_G(v)$, then the edge e contributes $+1$ in d_1 and $+1$ in d_2. Thus summing the degrees counts each edge twice. If $u = v$, then say d_2 does not exist, so we have the edge e contributing

2 in d_1. Hence in the summation $d_1 + d_2 + \ldots + d_n$ each edge of G, contributes $+2$. Since there are m edges then,

$$\begin{aligned} 2m &= d_1 + d_2 + \ldots + d_n \\ m &= \frac{1}{2}\sum_{i=1}^{n} d_i. \end{aligned}$$

□

We state the following theorem on counting the number of edges of the line graph of G without proof. However, we will prove an alternative proposition thereafter. We refer the reader to [3, 9].

Theorem 2.3.6. *Let G be a simple graph of order n and size m with degree sequence $d_1, d_2, \ldots, d_n$. Then the number of edges of the line graph $L(G)$ is given by,*

$$m(L(G)) = -m + \frac{1}{2}\sum_{i=1}^{i=n} (d_i)^2.$$

The following proposition is an alternative of Theorem 2.3.6. We refer the reader to [6, 9] for further details.

Proposition 2.3.7. *Let G be a graph of order n and size m and let $L(G)$ be the line graph of G. Let $d_1, d_2, \ldots, d_n$ be the degree sequence of the graph G. Then the size of $L(G)$ is,*

$$m(L(G)) = \sum_{i=1}^{n} \binom{d_i}{2}.$$

Proof. Consider a simple graph G with the degree sequence $d_1, d_2, \ldots, d_n$, then it follows that,

$$\begin{aligned}
m(L(G)) &= -m + \frac{1}{2}\sum_{i=1}^{i=n}(d_i)^2 \qquad \text{by Theorem 2.3.6} \\
&= -\frac{1}{2}\sum_{i=1}^{n} d_i + \frac{1}{2}\sum_{i=1}^{n}(d_i)^2 \qquad \text{by Lemma 2.3.5} \\
&= \frac{1}{2}\sum_{i=1}^{n}(d_i)^2 - \frac{1}{2}\sum_{i=1}^{n} d_i \\
&= \frac{1}{2}\sum_{i=1}^{n} d_i(d_i-1) \\
&= \sum_{i=1}^{n} \frac{d_i(d_i-1)(d_i-2)!}{2!(d_i-2)!} \\
&= \sum_{i=1}^{n} \frac{d_i!}{2!(d_i-2)!} \\
&= \sum_{i=1}^{n} \binom{d_i}{2}.
\end{aligned}$$

□

2.4 Line graphs of some classes of graphs

In this section, we give line graphs of some classes of graphs defined in Section 1.4.

2.4.1 Cycle graphs

Recall Definition 1.4.7, that a cycle graph of order n denoted by C_n, is a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{v_1v_2, v_2v_3, v_3v_4, \ldots, v_{n-1}v_n, v_nv_1\}$. The following proposition states without proof some properties of cycle graphs, see [3, 6].

Proposition 2.4.1. *Let C_n be a cycle graph of order n. Then,*

- *C_n is 2-regular.*

- *C_n is a connected graph.*

- *The sum of the degrees of the vertices of C_n is twice the number of vertices.*

We now state and prove a theorem on the line graph of a cycle graph. We refer the reader to [3].

Theorem 2.4.2. *Let C_n be a cycle graph of order n. Then the line graph $L(C_n)$ of C_n is a cycle graph of order n.*

Proof. We know by Definition 1.4.7, that a cycle graph C_n is a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{v_1v_2, v_2v_3, v_3v_4, \ldots, v_{n-1}v_n, v_nv_1\}$. By Definition 2.3.1, the vertex set of the line graph of C_n is given by,

$$\begin{aligned} V(L(C_n)) &= \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\} \\ &= \left\{e_{12}, e_{23}, \ldots, e_{(n-1)n}, e_{n1}\right\}. \end{aligned}$$

To obtain the edge set of the line graph $L(C_n)$, we relabel vertex e_{ij} by w_i where $i < j < n-1$, w_{n-1} for $e_{(n-1)n}$ and w_n for e_{n1}, thus $V(L(C_n)) = \{w_1, w_2, \ldots, w_n\}$. Note that since $e_{ij} = v_iv_j$, then we know that e_{ij} and e_{jk} will be an edge of $L(C_n)$. Hence the edge set is given by,

$$\begin{aligned} E(L(C_n)) &= \left\{e_{12}e_{23}, e_{23}e_{34}, \ldots, e_{(n-1)n}e_{n1}, e_{n1}e_{12}\right\} \\ &= \{w_1w_2, w_2w_3, \ldots\ w_{n-1}w_n, w_nw_1\}. \end{aligned}$$

This implies that $L(C_n)$ is a cycle graph of order n, by Definition 1.4.7. □

Example 2.4.3. The graphs in Figure 2.5 shows a cycle C_5, and illustrates the constructed line graph $L(C_5)$. The cycle graph C_5 has order $n = 5$, size $m = 5$ and degree sequence $2, 2, 2, 2, 2$. The Line graph of C_5, $L(C_5)$ has order 5 and size 5.

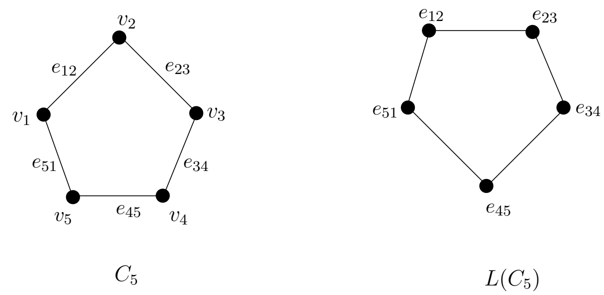


Figure 2.5: C_5 and $L(C_5)$.

We verify Theorem 2.3.6 for the size of $L(C_5)$, using size and degree sequence of C_5.

$$\begin{aligned}
|E[L(C_5)]| &= -m + \frac{1}{2}\sum_{i=1}^{i=n}(d_i)^2 \\
&= -5 + \frac{1}{2}(2^2 + 2^2 + 2^2 + 2^2 + 2^2) \\
&= -5 + \frac{1}{2}(20) \\
&= 5.
\end{aligned}$$

2.4.2 Trees in general

Recall Definition 1.4.1, that a tree is a connected graph with no cycles. The following proposition states without proof some properties of trees. We refer the reader to [3, 6, 9].

Proposition 2.4.4. *Let t_n be a tree on n vertices. Then,*

(a) t_n has $n-1$ edges.

(b) t_n has at least two vertices of degree 1.

(c) t_n has one unique path between every pair of distinct vertices.

(d) t_n has the degree sequence $d_1, d_2, \ldots, d_n$, for $n \geq 2$ iff $\sum_{i=1}^{i=n} d_i = 2n-2$.

(e) Every tree is a bipartite graph.

We now state and prove a proposition on the line graph of a tree.

Proposition 2.4.5. *Let t_n be a tree on n vertices. Then the line graph of t_n, $L(t_n)$ has $n-1$ vertices.*

Proof. By Proposition 2.4.4, t_n has $n-1$ edges. Hence by Definition 2.3.1, the vertex set of the line graph is the edge set of t_n. $\square$

Example 2.4.6. The graphs in Figure 2.6 shows a tree, t_{10} and illustrates the constructed line graph $L(t_{10})$. The tree, t_{10} has order $n = 10$, size $m = 9$ and degree sequence $1, 1, 1, 1, 1, 2, 2, 3, 3, 3$. The Line graph of t_{10}, $L(t_{10})$ has order 9 and size 11.

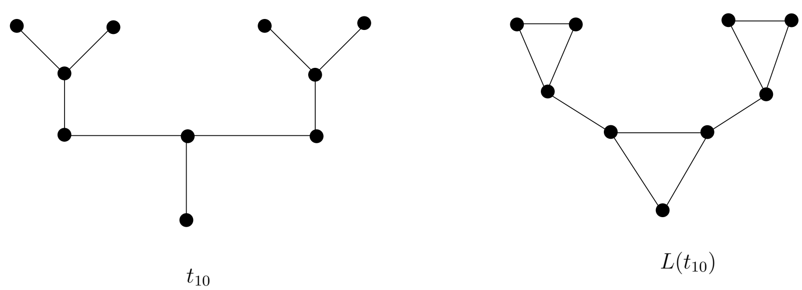


Figure 2.6: t_{10} and $L(t_{10})$.

2.4.3 Path trees

Recall Definition 1.4.1, that a path tree of order n is a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{v_i v_{i+1} \mid \text{for } \ i \in \{1, 2, \ldots, n-1\}\ \}$. The following proposition, states without proof a property of P_n. We refer the reader to [3] for further details.

Proposition 2.4.7. *Let P_n be a Path tree on n vertices. Then P_n is a tree with 2 vertices of degree 1 and $n-2$ vertices of degree 2.*

We now state a theorem of the line graph of a path tree P_n. We refer the reader to [3, 9] for further details.

Theorem 2.4.8. *Let P_n be a path tree on n vertices where $n \geq 2$. Then the line graph of P_n, $L(P_n)$ is a path tree P_{n-1}.*

Proof. We know by Definition 1.4.1, that a path tree of order n is a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{v_i v_{i+1} \mid \text{ for } \ i \in \{1, 2, \ldots, n-1\}\ \}$. By Definition 2.3.1, the vertex set of the line graph of P_n is given by,

$$\begin{aligned} V(L(P_n)) &= \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\} \\ &= \left\{e_{12}, e_{23}, \ldots, e_{(n-1)n}\right\}. \end{aligned}$$

To obtain the edge set of the line graph $L(P_n)$, we relabel vertex e_{ij} by w_i where $i < j$ and w_{n-1} for $e_{(n-1)n}$, thus $V(L(P_n)) = \{w_1, w_2, \ldots, w_{n-1}\}$. Hence the edge set is given by,

$$\begin{aligned} E(L(P_n)) &= \{e_{12}e_{23}, e_{23}e_{34}, \ldots, e_{n-2}e_{n-1}, e_{n-1}e_n\} \\ &= \{w_1w_2, w_2w_3, \ldots\ w_{n-2}w_{n-1}\}. \end{aligned}$$

This implies that $L(P_n)$ is a path tree of order $n-1$, P_{n-1}. □

Theorem 2.4.9. *The line graph of a simple graph G is a path tree, if and only if G is a path tree.*

Proof. Let G be a path tree P_n on n vertices. Then by Theorem 2.4.8, the line graph $L(P_n)$ is a path tree, P_{n-1}.

Now let $L(G)$ be a path tree. Thus the degree of each vertex of $L(G)$ is either 2 or 1, so no vertex has degree greater than 2. Hence G must either be a cycle graph or a path graph, however it cannot be a cycle graph because by Theorem 2.4.2 the line graph of a cycle is a cycle. Thus G is a path tree. □

Example 2.4.10. The graphs in Figure 2.7 shows a path tree P_5 and illustrates the constructed line graph $L(P_5)$. The tree, P_5 has order $n = 5$, size $m = 4$ and degree sequence $1, 1, 2, 2, 2$. The Line graph of P_5, $L(P_5)$ has order 4 and size 3.

Figure 2.7: P_5 and $L(P_5)$.

2.4.4 Star trees

Recall Definition 1.4.1, that a star graph of order n is a graph with vertex set $V(S_n) = \{v_1, v_2, \ldots, v_{n-1}, x\}$ and edge set $E(S_n) = \{xv_1, xv_2, \ldots, xv_{n-1}\}$. The following proposition states without proof some properties of star trees, see [5, 17].

Proposition 2.4.11. *Let S_n be a star tree. Then S_n has $n-1$ vertices of degree 1 and one vertex of degree $n-1$.*

We now state and prove a property of the line graph of a star tree.

Proposition 2.4.12. *Let S_n be a star tree. Then the line graph of S_n, $L(S_n)$ is a complete graph of order $n-1$.*

Proof. We know by Definition 1.4.1, a star graph is a graph with vertex set $V(S_n) = \{v_1, v_2, \ldots, v_{n-1}, x\}$ and edge set $E(S_n) = \{xv_1, xv_2, \ldots, xv_{n-1}\}$. By Definition 2.3.1, the vertex set of the line graph of S_n is given by,

$$\begin{aligned} V(L(S_n)) &= \{xv_1, xv_2, \ldots, xv_{n-1}\} \\ &= \left\{e_{x1}, e_{x2}, \ldots, e_{x(n-1)}\right\}. \end{aligned}$$

To obtain the edge set of the line graph $L(S_n)$, we relabel vertex e_{xi} by w_i, thus $V(L(S_n)) = \{w_1, w_2, \ldots, w_{n-1}\}$. Hence, the edge set is given by $E(L(S_n)) = \{w_i w_j \mid i \neq j \text{ and } i, j \in \{1, 2, \ldots, n-1\}\}$. By Definition 1.4.5 of a complete graph, $L(S_n)$ is a complete graph of order $n-1$. □

Example 2.4.13. The graphs in Figure 2.8 shows a star graph S_8 and illustrates the constructed line graph $L(S_8)$. The star graph, S_8 has order $n = 8$, size $m = 7$ and degree sequence $1, 1, 1, 1, 1, 1, 1, 7$. The Line graph of S_8, $L(S_8)$ is a complete graph of order 7 and is 6-regular.

Figure 2.8: S_8 and $L(S_8)$.

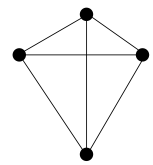
2.4.5 Complete graphs

Recall Definition 1.4.5, that a complete graph is a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{(v_i v_j) \mid i \neq j \; i, j \in \{1, 2, \ldots, n\}\}$. The following proposition states without proof some properties of complete graphs. We refer the reader to [3, 6] for further details.

Proposition 2.4.14. *Let K_n be a complete graph of order n. Then,*

- *K_n is $(n-1)$-regular.*
- *Every two vertices are adjacent.*
- *The number of edges is given by $n(n-1)/2$.*

Example 2.4.15. The graphs in Figure 2.9 shows a complete graph K_4 and illustrates the constructed line graph $L(K_4)$. The complete graph K_4 has order $n = 4$, size $m = 6$ and degree sequence $3, 3, 3, 3$. The Line graph of K_4, $L(K_4)$ has order 6 and size 12.



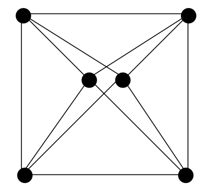


Figure 2.9: K_4 and $L(K_4)$.

2.5 Conclusion

In this chapter, we defined the line graph of a graph. We gave a few examples and properties of a line graph of a graph. Using the classes of graphs we stated in Chapter 1, we gave line graphs of some of these classes of graphs. We were able to state and prove some properties of line graphs for these classes of graphs.

Chapter 3

Triangle graph of a graph

3.1 Introduction

In this chapter, we discuss the concept of a triangle graph of a graph. We begin by giving a formal definition of a triangle graph of a graph and illustrate with examples. Then, we give some properties of triangle graphs. Thereafter, we give some triangle graphs of some classes of graphs.

In this chapter to the end of the dissertation, a graph will mean a simple graph, that is a graph with no parallel edges and no loops.

3.2 Basic definition

In this section, we begin by defining some notations and then giving a formal definition of a triangle of a graph. In addition, we mention, the concept of the cycle graph of a graph and illustrate using examples the difference between a cycle graph of a graph and the new defined, triangle graph of a graph.

Definition 3.2.1. A triangle, is a subgraph of G isomorphic to C_3. Thus a triangle is a subgraph with vertex set $\{v_i, v_j, v_k\}$ such that v_i, v_j and v_k are distinct vertices and edge set $\{v_iv_j, v_iv_k, v_jv_k\}$.

Notations:

Let G be a graph of order n with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Let $v_iv_j \in E(G)$, we denoted v_iv_j by $e_{ij} = e_{ji}$. A triangle on vertex set $\{v_i, v_j, v_k\}$ will be denoted by $t_{ijk} = t_{ikj} = t_{jki} = t_{jik} = t_{kij} = t_{kji}$. We denote the set of triangles of G by $T(G)$.

We are now in the position to define the triangle graph of a graph.

Definition 3.2.2. Let G be a graph. We define the triangle graph of G, to be a graph denoted by, $T_\Delta(G)$ with vertex set $V(T_\Delta(G)) = T(G) = \{t_{ijk} \mid e_{ij}, e_{jk}, e_{ik} \in E(G)\ \}$ and edge set $E(T_\Delta(G)) = \{t_{ijk}t_{lmn} \mid \{i,j,k\} \cap \{l,m,n\} \neq \emptyset$ and $t_{ijk}, t_{lmn} \in T(G)\ \}$.

Note that the triangle graph of a graph is not the same as the cycle graph of a graph, where the cycle is of order 3. For cycle graphs of a graph, we refer the reader to [13], but we give a formal definition of a cycle graph of a graph, and illustrate with an example the difference with triangle graph of a graph.

Definition 3.2.3. Let G be a graph. We define the cycle graph of a graph to be a graph denoted by $C(G)$, where the vertices are the chordless cycles of G and two vertices are said to be adjacent if their corresponding chordless cycles share a common edge.

Example 3.2.4. The graph G_1 given in Figure 3.1 has vertex set
$V(G_1) = \{v_1, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$ and edge set
$E(G_1) = \{e_{14}, e_{15}, e_{34}, e_{45}, e_{57}, e_{59}, e_{67}, e_{68}, e_{78}, e_{79}\}$. The set of triangles of G_1, is $T(G_1) = \{v_1v_4v_5, v_5v_7v_9, v_6v_7v_8\} = \{t_{145}, t_{579}, t_{678}\}$. The triangle graph of G_1 has vertex set, $V(T_\Delta(G_1)) = T(G_1) = \{t_{145}, t_{579}, t_{678}\}$ and has edge set $E(T_\Delta(G_1)) = \{t_{145}t_{579}, t_{579}t_{678}\}$.

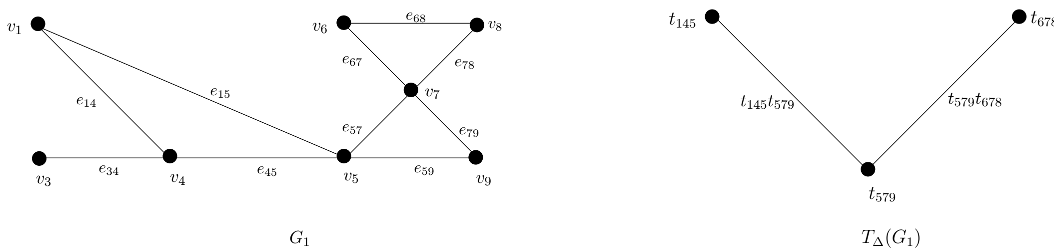


Figure 3.1: Graph G_1 and the triangle graph of G_1.

We now construct the cycle graph of G_1 where the cycles are of order 3. The cycle graph will have vertex set $\{t_{145}, t_{678}, t_{679}\}$. We note that there is no edge intersection between t_{145} and t_{579}, t_{145} and t_{678} and t_{579} and t_{678}. Hence $E[C(G)] = \{\}$.

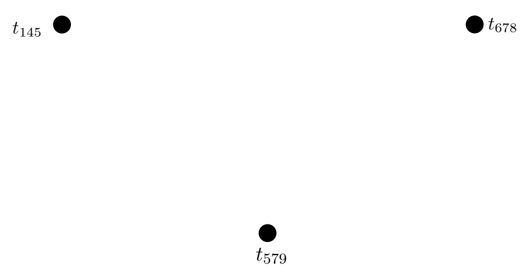


Figure 3.2: The cycle graph of G_1.

Example 3.2.5. The graph G_2 given in Figure 3.3 has vertex set $V(G_2) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$ and edge set $E(G_2) = \{e_{12}, e_{13}, e_{23}, e_{24}, e_{34}, e_{35}, e_{45}, e_{46}, e_{56}, e_{67}, e_{68}, e_{78}, e_{79}, e_{8,10}, e_{9,10}\}$. The set of triangles of G_2, $T(G_2) = \{v_1v_2v_3, v_2v_3v_4, v_3v_4v_5, v_4v_5v_6, v_6v_7v_8\} = \{t_{123}, t_{234}, t_{345}, t_{456}, t_{678}\}$. The triangle graph of G_2 has vertex set $V(T_\Delta(G_2)) = T(G_2) = \{t_{123}, t_{234}, t_{345}, t_{456}, t_{678}\}$ and has edge set $E(T_\Delta(G_2)) = \{t_{123}t_{234}, t_{123}t_{345}, t_{234}t_{345}, t_{234}t_{456}, t_{345}t_{456}, t_{456}t_{678}\}$.

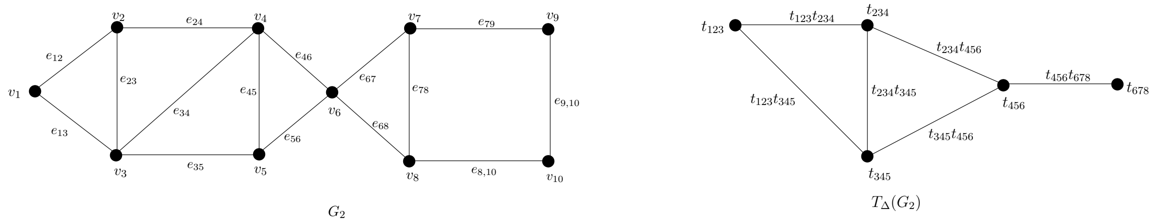


Figure 3.3: Graph G_2 and the triangle graph of G_2.

We now construct the cycle graph of G_2 where the cycles are of order 3 only. The cycle graph will have vertex set $\{t_{123}, t_{234}, t_{345}, t_{456}, t_{678}\}$ and edge set $\{t_{123}t_{234}, t_{234}t_{345}, t_{345}t_{456}\}$.

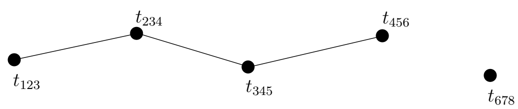


Figure 3.4: The cycle graph of G_2.

3.2.1 Basic properties

We now state and prove some basic properties of a triangle graph of a graph.
We recall Definition 1.3.2 that a graph is simple if it has no loops and multiple edges. Thus if G is a simple graph, it implies that there is at most one edge between each pair of distinct vertices.

Proposition 3.2.6. *Let G be a graph and let $T_\Delta(G)$ be the triangle graph of G. Then,*

(a) $T_\Delta(G)$ is a simple graph.

(b) $T_\Delta(G)$ is an empty graph of order 0 if G is a tree of any order.

Proof. (a) By Definition 3.2.2, of a triangle graph, $T_\Delta(G)$ is a simple graph since there is a single edge connecting a pair of distinct triangles whose vertex sets has a non-empty intersection.

(b) Let G be a tree of any order. By Definition 1.4.1, a tree has no cycles, which implies that G does not contain any set of triangles. Thus $T(G) = \{\} = \emptyset$, hence $T_{\Delta}(G)$ is an empty graph.

$\square$

Proposition 3.2.7. *Let G be a graph and $T_{\Delta}(G)$ the triangle graph of G. $T_{\Delta}(G)$ is empty if and only if G has no subgraph isomorphic to C_3.*

Proof. Assume $T_{\Delta}(G)$ is an empty graph of order 0. Then $V(T_{\Delta}(G)) = \{\}$. Thus $T(G) = \{\}$. This implies, the set of triangles of G is empty. Hence G has no cycles of order 3.
Assume that the graph G has no cycles of order 3. Then the set of triangles $T(G) = \{\}$. This implies $V(T_{\Delta}(G)) = \{\}$. Therefore, $T_{\Delta}(G)$ has order 0. $\square$

Recall that t_{ijk} is a triangle with vertex set $\{v_i, v_j, v_k\}$ which we now denote by $V(t_{ijk})$. Recall that $T(G)$ is the set of all triangles of G. We denote the union of the set of vertices of all triangles of G by $V(T(G)) = \{v_i \mid v_i \in V(t_{ijk}) \text{ where } \ t_{ijk} \in T(G)\ \}$.

Proposition 3.2.8. *If $V(T(G))$ can be partitioned into $V(T_1(G))$ and $V(T_2(G))$ where $T_1(G) \subset T(G)$ and $T_2(G) \subset T(G)$, then $T_{\Delta}(G)$ is a disconnected graph.*

Proof. Let $T_1(G) \subset V(T_{\Delta}(G))$, $T_2(G) \subset V(T_{\Delta}(G))$. Let $T_1(G)$ be the set of triangles of G given by

$$T_1(G) = \{v_iv_jv_k \mid v_iv_j, v_iv_k, v_jv_k \in E(G)\ \}$$

and let $T_2(G)$ be the set of triangles of G given by

$$T_2(G) = \{v_xv_yv_z \mid v_xv_y, v_xv_z, v_yv_z \in E(G)\ \}$$

where $\{i, j, k\} \cap \{x, y, z\} = \emptyset$. Hence we have that

$$V(T_1(G)) = \{v_i \mid v_i \in V(t_{ijk})\ \forall\ t_{ijk} \in T_1(G)\ \}$$

and $V(T_2(G)) = \{v_x \mid v_x \in V(t_{xyz})\ \forall\ t_{xyz} \in T_2(G)\ \}$. Hence by definition of the triangle graph, there will be no edges between any vertex of $T_{\Delta}(G)$ in $T_1(G)$ and a vertex $T_2(G)$. By Definition 1.3.9, the graph $T_\Delta(G)$ is disconnected since there exist no path between the vertices in $T_1(G)$ and $T_2(G)$ in the construction of $T_\Delta(G)$. $\square$

Corollary 3.2.9. *Let G be a graph of order n, $V(T(G))$ be the vertices of all triangles of G and let $T_i(G) \subset T(G)$. If $V(T(G))$ can be partitioned into q sets, $V(T_1(G)), V(T_2(G)), \ldots, V(T_q(G))$, then the triangle graph $T_\Delta(G)$ is a disconnected graph with q components.*

Example 3.2.10. Let G be the graph given in Figure 3.5. Then $V(T(G)) = \{t_{234}, t_{678}, t_{579}\}$ can be partitioned into $V(T_1(G))$ and $V(T_2(G))$ as follows, $V(T_1(G)) = \{t_{234}\}$ and $V(T_2(G)) = \{t_{678}, t_{579}\}$. The triangle graph $T_\Delta(G)$ is shown in the same Figure 3.5 as a disconnected graph with 2 components.

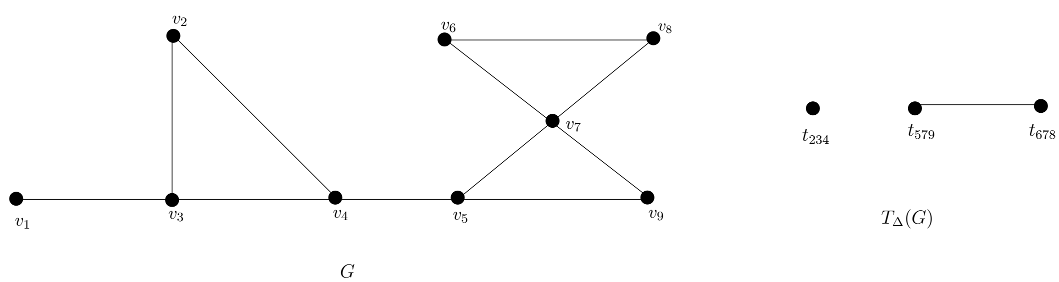


Figure 3.5: Graph G and the triangle graph $T_\Delta(G)$.

3.3 Triangle graphs of certain classes of graphs

In this section, we discuss triangle graphs of certain classes of graphs.

3.3.1 Triangle graph of a wheel graph

Recall Definition 1.4.11 that a wheel graph denoted W_n, is a graph with vertex set $V = \{v_1, v_2, \ldots, v_{n-1}, v_x\}$ and edge set

$$E = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-2}, v_{n-1}\}, \{v_{n-1}, v_1\}, \{v_1, v_x\}, \{v_2, v_x\}, \ldots, \{v_{n-1}, v_x\}\}.$$

It is clear that an edge set of the form $\{v_iv_{i+1}, v_iv_x, v_{i+1}v_x\}$, for $i \in \{1, 2, \ldots, n-1\}$ gives a triangle in W_n. Note that when $i = n-1$ then $i+1 = 1$. The following proposition summarizes a few properties on wheel graphs.

Proposition 3.3.1. *Let W_n be a wheel graph. Then,*

- *vertex v_x will have degree $n-1$.*
- *the number of edges is given by $2(n-1)$.*
- *the number of triangles in W_n is $n-1$ if $n > 4$.*
- *the number of triangles in W_n is n if $n = 4$.*
- *W_n is a planar graph.*

We are now in a position to state and prove a proposition on the triangle graph of a wheel graph. We note that by definition of a wheel graph, W_n is a simple graph if $n \geq 4$. Thus we only state the proposition when $n \geq 4$.

Proposition 3.3.2. *Let W_n be a wheel graph and let $T_\Delta(W_n)$ be the triangle graph of W_n. Then $T_\Delta(W_n)$ is a complete graph K_n if $n = 4$.*

Proof. The graph W_4 is given in Figure 3.6. Thus $T(W_4) = \{v_1v_2v_x, v_2v_3v_x, v_3v_1v_x, v_1v_2v_3\} = \{z_1, z_2, z_3, z_4\}$, respectively. Hence $V(T_\Delta(W_4)) = \{z_1, z_2, z_3, z_4\}$. But v_x is a vertex of triangles z_1, z_2 and z_3, hence z_1z_2, z_2z_3 and z_3z_1 are edges of $T_\Delta(W_4)$. In addition, triangle $v_1v_2v_3 = z_4$ has a common vertex with all other three triangles z_1 z_2 and z_3. Thus z_1z_4, z_2z_4 and z_3z_4 are edges of $T_\Delta(W_4)$. Thus we have $T_\Delta(W_4)$ a graph on

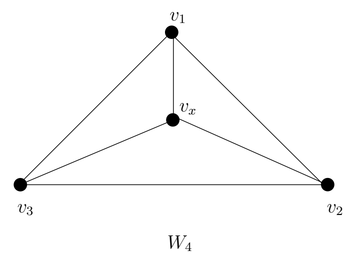


Figure 3.6: The wheel graph, W_4.

vertex set $\{z_1, z_2, z_3, z_4\}$ and edge set $\{z_i z_j | i \neq j \text{ and } i, j \in \{1, 2, 3, 4\}\}$ which is a complete graph K_4.

$\square$

Proposition 3.3.3. *Let W_n be a wheel graph and let $T_\Delta(W_n)$ be the triangle graph of W_n. Then $T_\Delta(W_n)$ is a complete graph K_{n-1} if $n > 4$.*

Proof. By the recalled definition of a wheel graph, it is clear that, a subgraph of W_n with edge set $\{v_i v_{i+1}, v_{i+1} v_x, v_i v_x\}$ is a triangle. We denote this triangle by $t_{i(i+1)x}$ and we label $t_{i(i+1)x}$ by z_i. Thus

$$\begin{aligned} T(W_n) &= \{t_{i(i+1)x} \mid i \in \{1, 2, \ldots, n-1\} \text{ and } i+1 = 1 \text{ if } i = n\ \} \\ &= \{z_1, z_2, \ldots, z_{n-1}\} = V(T_\Delta(W_n)). \end{aligned}$$

But vertex v_x is in all the triangles z_i. Hence there is an edge between all pairs z_i, z_j in $V(T_\Delta(W_n))$. Thus edge set $E(T_\Delta(W_n)) = \{z_i z_j \mid i \neq j, i, j \in \{1, 2, \ldots, n-1\}\ \}$. Thus, $T_\Delta(W_n)$ is a complete graph of order $n-1$. $\square$

Corollary 3.3.4. *Let W_n be a wheel graph and let $T_\Delta(W_n)$ be the triangle graph of W_n. Then the triangle graph $T_\Delta(W_n)$ is $n-2$ regular.*

Corollary 3.3.5. *Let W_n be a wheel graph and let $T_\Delta(W_n)$ be a triangle graph of W_n. Then the number of edges of $T_\Delta(W_n)$ is given by $\frac{(n-1)(n-2)}{2}$.*

3.3.2 Triangle graph of a Helm Graph

In this subsection, we define a special class of graphs called helm graphs which are derived from wheel graphs.

Definition 3.3.6. A *Helm* graph, denoted by H_n is a graph obtained from a wheel graph W_n, by adding $n-1$ edges and $n-1$ vertices such that the vertex set is $V = \{v_1, v_2, \dots, v_{n-1}, v_x, w_1, w_2, \dots, w_{n-1}\}$ and edge set

$$\begin{aligned} E &= \{\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-2}, v_{n-1}\}, \{v_{n-1}, v_1\}, \{v_1, v_x\}, \{v_2, v_x\}, \dots, \{v_{n-1}, v_x\}\} \\ &\quad \cup \{v_1 w_1, v_2 w_2, v_3 w_3, \dots, v_{n-1} w_{n-1}\}. \end{aligned}$$

For further details on helm graphs, we refer the reader to [21].

It is clear that the triangles of the wheel graph W_n are the triangles of the new graph H_n. The following proposition summarizes a few properties on helm graphs.

Proposition 3.3.7. *Let H_n be a helm graph. Then,*

- *H_n is connected.*
- *vertex v_x has degree $n-1$.*
- *the order of H_n is $2n-1$ and the size is $3(n-1)$.*
- *the number of triangles in H_n is $n-1$.*
- *H_n is a planar graph.*

We are now in a position to state and prove a proposition on the triangle graph of a helm graph.

Proposition 3.3.8. *Let H_n be a helm graph of order $2n-1$ and let $T_\Delta(H_n)$ be the triangle graph of H_n. Then $T_\Delta(H_n)$ is a complete graph K_{n-1}.*

Proof. For this proof we apply the same argument as in Proposition 3.3.3 on wheel graphs. □

Corollary 3.3.9. *Let H_n be a helm graph of order $2n-1$ and let $T_\Delta(H_n)$ be the triangle graph of H_n. Then triangle graph $T_\Delta(H_n)$ is $n-2$ regular.*

Example 3.3.10. The graphs in Figure 3.7 are an example of a helm graph, H_4 and its triangle graph $T_\Delta(H_4)$.

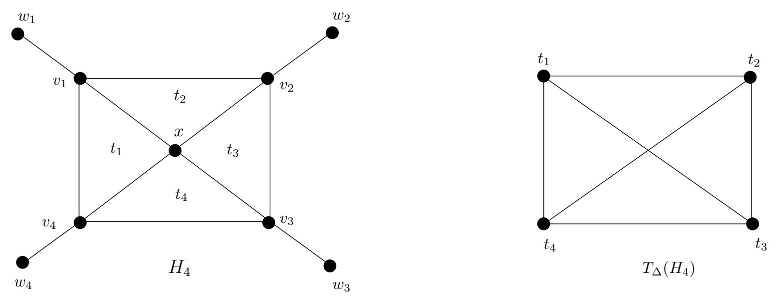


Figure 3.7: H_4 and $T_\Delta(H_4)$.

3.3.3 Triangle graph of a Sunflower graph

In this subsection, we define a special class of flower graphs called sunflower graphs. For the definition of a flower graph we refer the reader to Chapter 1, Definition 1.4.13.

Definition 3.3.11. A *sunflower* graph denoted by $S_{n\times 3}$, is a flower graph where $m = 3$ and $n \geq 2$ with vertex set $V = \{v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n\}$ and edge set $E = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\} \cup \{v_1w_1, v_1w_2, v_2w_2, \ldots, v_{n-1}w_n, v_nw_n, v_nw_1\}$.

By Definition 3.2.1 of a triangle, it is clear that the edge set of the form $\{v_iv_{i+1}, v_{i+1}w_i, v_iw_i\}$ for $i \in \{1, 2, \ldots, n\}$ and $i+1 = 1$ if $i = n$, is an edge set of a triangle of $S_{n\times 3}$. Thus the vertex set for this triangle is $\{v_i, v_{i+1}, w_i\}$ and hence the triangle will be denoted by $t_{ii(i+1)} = t_i$.

Proposition 3.3.12. *Let $S_{n\times 3}$ be a sunflower graph. Then,*

- *$S_{n\times 3}$ is a connected graph.*

- *the order of $S_{n\times 3}$ is $2n$ and the size is $3n$.*

- *the degree sequence of vertices is given by $\underbrace{2,2,2\ldots,2}_{n}, \underbrace{4,4,4,4}_{n}$.*

There are three different cases for triangle graphs of sunflower graphs to be considered namely; $n = 2$, $n = 3$ and $n > 3$.

Theorem 3.3.13. *Let $G = S_{2\times 3}$ be a sunflower graph and let $T_\Delta(G)$ be the triangle graph of $S_{2\times 3}$. Then $T_\Delta(G)$ is a path graph P_2.*

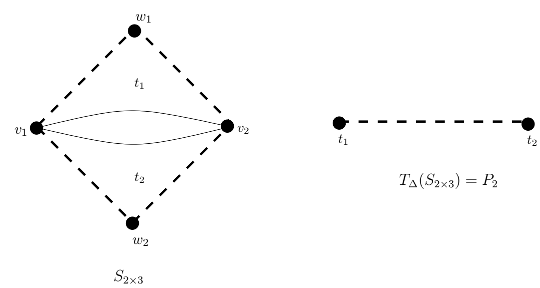


Figure 3.8: $S_{2\times 3}$ and $T_\Delta(S_{2\times 3})$.

Proof. The graph $G = S_{2\times 3}$ is given in Figure 3.8. Then the set of triangles of G, $T(G) = \{v_1v_2w_1, v_1v_2w_2\} = \{t_{112}, t_{221}\} = \{t_1, t_2\}$. Hence $V(T_\Delta(G)) = \{t_{112}, t_{221}\}$. The edge set $E(T_\Delta(G)) = \{t_{112}t_{221}\} = \{t_1t_2\}$ since $\{1, 1, 2\} \cap \{2, 2, 1\}$ is not empty. Hence, $T_\Delta(G)$ is a path graph P_2. $\square$

We now consider the sunflower graph $S_{3\times 3}$.

Theorem 3.3.14. *Let $G = S_{3\times 3}$ be a sunflower graph and let $T_\Delta(G)$ be a triangle graph of $S_{3\times 3}$. Then $T_\Delta(G)$ is a complete graph K_4.*

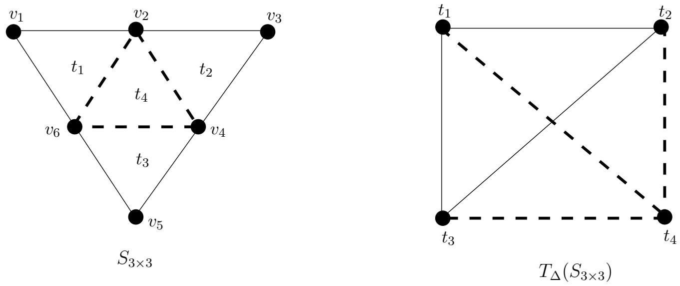


Figure 3.9: $S_{3\times3}$ and $T_\Delta(S_{3\times3})$.

Proof. The graph $G = S_{3\times3}$ is given in Figure 3.9. Then the set of triangles of G, $T(G) = \{v_1v_2v_6, v_2v_3v_4, v_4v_5v_6, v_2v_4v_6\} = \{t_1, t_2, t_3, t_4\} = \{V(T_\Delta(G))\}$. It is clear that each triangle t_i has a common vertex with all the other triangles. Hence, $T_\Delta(G)$ is a complete graph K_4. □

We now consider the sunflower graph $S_{n\times3}$ for $n > 3$.

Theorem 3.3.15. *Let $G = S_{n\times3}$ be a sunflower graph for $n > 3$ and let $T_\Delta(G)$ be the triangle graph of $S_{n\times3}$. Then $T_\Delta(G)$ is a cycle graph C_n.*

Proof. By Definition 3.3.11 of a sunflower graph, the triangles of G are denoted by $t_{ii(i+1)} = t_i$ where $i+1 = 1$ if $i = n$ and $i \in \{1, 2, \ldots, n\}$. Thus

$$\begin{aligned} V(T_\Delta(G)) &= \{t_{ii(i+1)} \mid i \in 1, 2, \ldots, n \text{ and } i+1 = 1 \text{ if } i = n\ \} \\ &= \{t_{112}, t_{223}, t_{334}, \ldots, t_{nn1}\} \\ &= \{t_1, t_2, t_3, \ldots, t_n\}. \end{aligned}$$

By definition, $\{i, i, i+1\} \cap \{i+1, i+1, i+2\} \neq \emptyset$ for defined i. Thus

$$\begin{aligned} E(T_\Delta(G)) &= \{t_{ii(i+1)}t_{(i+1),(i+1),(i+2)} \mid i \in 1, 2, \ldots, n \text{ and } i+1 = 1 \text{ if } i = n\ \} \\ &= \{t_it_{i+1} \mid i \in 1, 2, \ldots, n \text{ and } i+1 = 1 \text{ if } i = n\ \} \\ &= \{t_1t_2, t_2t_3, t_3t_4, \ldots, t_{n-1}t_n, t_nt_1\}. \end{aligned}$$

Hence, $T_\Delta(G)$ is a cycle graph C_n. □

Example 3.3.16. Consider $S_{4\times 3}$ given in Figure 3.10. The set of triangles $T(S_{4\times 3}) = \{t_1, t_2, t_3, t_4\}$. Hence, the set of vertices of the triangle graph, $V(T_\Delta(S_{4\times 3})) = \{t_1, t_2, t_3, t_4\}$. The edge set of the triangle graph, $E(T_\Delta(S_{4\times 3})) = \{t_1t_2, t_2t_3, t_3t_4, t_1t_4\}$. Hence, $T_\Delta(S_{4\times 3}) = C_4$ verifying Theorem 3.3.15.

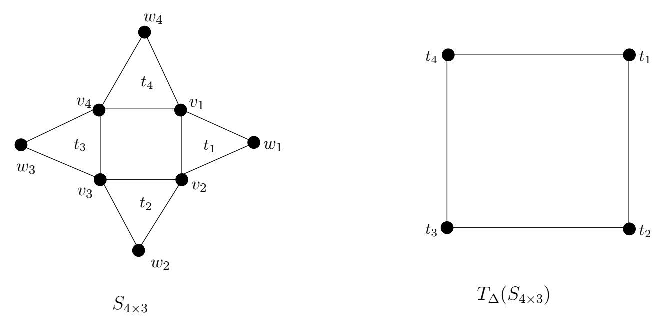


Figure 3.10: $S_{4\times 3}$ and $T_\Delta(S_{4\times 3})$.

3.3.4 Triangle graph of a fan graph

Definition 3.3.17. A *fan* graph denoted by f_n, is a graph with vertex set
$V = \{v_1, v_2, \ldots, v_{n-1}, v_x\}$ and edge set
$E = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-2}, v_{n-1}\}, \{v_1, v_x\}, \{v_2, v_x\}, \ldots, \{v_{n-1}, v_x\}\}.$

By Definition 3.2.1 of a triangle, it is clear that the edge set of the form $\{v_iv_{i+1}, v_iv_x, v_{i+1}v_x\}$ for $i \in \{1, 2, \ldots, n-2\}$, gives a triangle in f_n and we denote this triangle by $t_{i(i+1)x}$. For further details on fan graphs, we refer the reader to [15, 18]. We now state some basic properties of fan graphs.

Proposition 3.3.18. *Let f_n be a fan graph. Then,*

- *f_n is a connected graph.*
- *vertex v_x has degree $n-1$.*

- *the order of f_n is $n+1$ and the size is $2n-3$.*

- *the number of triangles in f_n is $n-2$.*

We are now in a position to state and prove a theorem on the triangle graph of a fan graph.

Theorem 3.3.19. *Let f_n be a fan graph and let $T_\Delta(f_n)$ be a triangle graph of f_n. Then $T_\Delta(f_n)$ is a complete graph K_{n-2}.*

Proof. By Definition 3.3.17 of a fan, the triangles have edge sets of the form $\{v_iv_{i+1}, v_iv_x, v_{i+1}v_x\}$ for $i \in \{1, 2, \ldots, n-2\}$ denoted by $t_{i(i+1)x}$. Let triangle $t_{i(i+1)x}$ be labeled t_i. Then $T(f_n) = \{t_1, t_2, \ldots, t_{n-2}\} = \{V(T_\Delta(f_n))\}$. But each $t_{i(i+1)x}$ has vertex v_x, thus $E(T_\Delta(f_n)) = \{t_it_j \mid i \neq j \text{ and } i, j \in \{1, 2, \ldots, n-2\} \}$. Hence, $T_\Delta(f_n)$ is a complete graph of order $n-2$. □

Example 3.3.20. Consider the fan graph, f_4 given in Figure 3.11. The set of triangles, $T(f_4) = \{t_1, t_2, t_3\}$. Thus, the vertex set of the triangle graph, $V(T_\Delta(f_4)) = \{t_1, t_2, t_3\}$. But v_x is a vertex of each triangle, thus the edge set of the triangle graph, $E(T_\Delta(f_4)) = \{t_1t_2, t_2t_3, t_1t_3\}$. Hence, $T_\Delta(f_4) = K_3$ verifying Theorem 3.3.19.

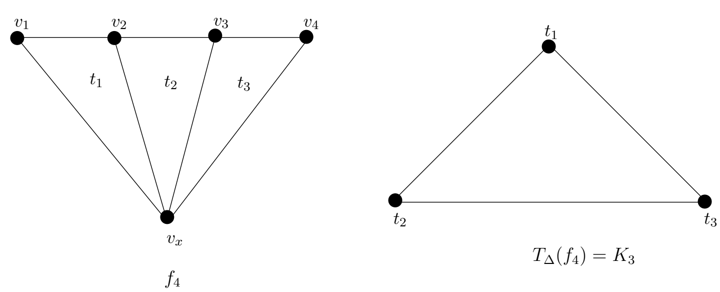


Figure 3.11: f_4 and $T_\Delta(f_4)$.

Corollary 3.3.21. *Let f_n be a fan graph and let $T_\Delta(f_n)$ be the triangle graph of f_n. Then triangle graph $T_\Delta(f_n)$ is $n-3$ regular.*

3.3.5 Triangle graph of a Friendship graph

Definition 3.3.22. A *friendship* graph denoted by F_n, is a graph with vertex set $V = \{v_1, v_2, \ldots, v_{2n}, v_x\}$ and edge set
$E = \{\{v_1, v_2\}, \{v_3, v_4\}, \ldots, \{v_{2n-1}, v_{2n}\}, \{v_1, v_x\}, \{v_2, v_x\}, \ldots, \{v_{2n}, v_x\}\}$.

By Definition 3.2.1 of a triangle, it is clear that the edge set of the form $\{v_i v_{i+1}, v_i v_x, v_{i+1} v_x\}$ for $i \in \{1, 3, 5, \ldots, 2n-1\}$, gives a triangle in F_n. We now state some basic properties of friendship graphs. For further detail on friendship graphs, we refer the reader to [18].

Proposition 3.3.23. *Let F_n be a friendship graph. Then,*

- *F_n is a connected graph.*
- *the order of F_n is $2n+1$ and the size is $3n$.*
- *the number of triangles in F_n is n.*
- *F_n is a planar graph.*

We are now in a position to state and prove a theorem on the triangle graph of a friendship graph.

Theorem 3.3.24. *Let F_n be a friendship graph and let $T_\Delta(F_n)$ be the triangle graph of F_n. Then $T_\Delta(F_n)$ is a complete graph K_n.*

Proof. Without loss of generality, we illustrate the proof using F_6 in the diagram in Figure 3.12.
The triangles in F_6 are denoted by t_i, thus we have the following set of triangles of F_6, $T(F_6) = \{t_1, t_2, t_3, t_4, t_5, t_6\}$. This implies that $V(T_\Delta(F_6)) = \{t_1, t_2, t_3, t_4, t_5, t_6\}$, the vertex set of $T_\Delta(F_6)$. Since all triangles in F_6 share a common vertex v_x, it implies that each vertex of $T_\Delta(F_6)$ is connected to all the other vertices. This implies that the edge set of $T_\Delta(F_6)$ is $E(T_\Delta(F_6)) = \{t_1t_2, t_1t_3, \ldots, t_1t_n, t_2t_3, \ldots, t_2t_n, t_3t_4, \ldots, t_3t_n, t_4t_5, t_4t_6, t_5t_6, t_1t_6\}$.
Hence, $T_\Delta(F_n) = T_\Delta(F_6)$ is a complete graph $K_n = K_6$. □

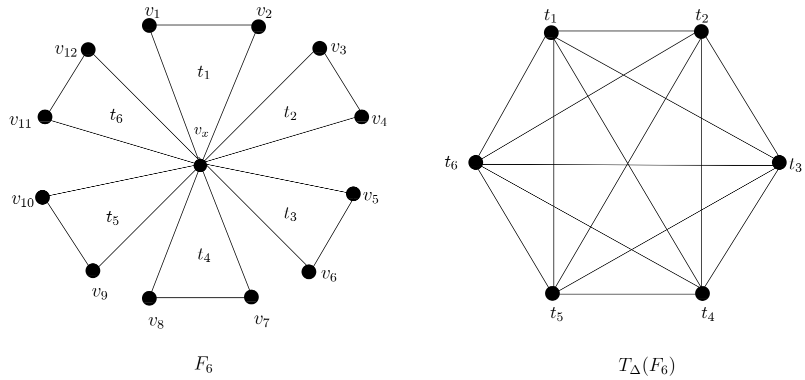


Figure 3.12: Illustration for the proof of Theorem 3.3.24.

3.4 Conclusion

In this chapter, we defined the triangle graph of a graph. We introduced some notations which we used throughout the chapter. We clarified the difference between the triangle graph of a graph and the cycle graph of a graph, where the cycle is of order 3. Thereafter, we stated and proved some basic properties of a triangle graph of a graph. We then discussed triangle graphs of certain classes of graphs, namely; helm graphs, sunflower graphs, fan graphs and friendship graphs.

Chapter 4

Triangle graph of a vertex-join

4.1 Introduction

In this chapter, we begin by defining the vertex-join of a graph, then we give some properties of triangle graphs of vertex-join of graphs. Finally, we discuss some well known classes of graphs which can be defined as vertex joins of other classes. Note that we only discuss simple graphs in this chapter.

4.2 Basic definition

In this section, we give a formal definition of a vertex-join of a graph, followed by illustrations of this concept through examples.

Definition 4.2.1. Let G be a graph of order n with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The vertex-join of G is the graph denoted $\hat{G}$ with vertex set $V(\hat{G}) = V \cup \{v_x\}$, where $x \notin \{1, \ldots, n\}$ and edge set $E(\hat{G}) = E(G) \cup \{v_1v_x, v_2v_x, \ldots, v_nv_x\}$. Thus, the vertex-join of a graph G is a graph of order $n + 1$.

Example 4.2.2. The graph G given in Figure 4.1 has vertex set $V(G) = \{v_1, v_2, v_3, v_4\}$ and edge set $E(G) = \{v_1v_2, v_2v_4, v_3v_4, v_1v_3\}$. The vertex-join of graph G has vertex set $V(\hat{G}) = \{v_1, v_2, v_3, v_4\} \cup \{v_x\}$ and edge set

$E(\hat{G}) = \{v_1v_2, v_1v_3, v_2v_4, v_3v_4\} \cup \{v_1v_x, v_2v_x, v_3v_x, v_4v_x\}$.

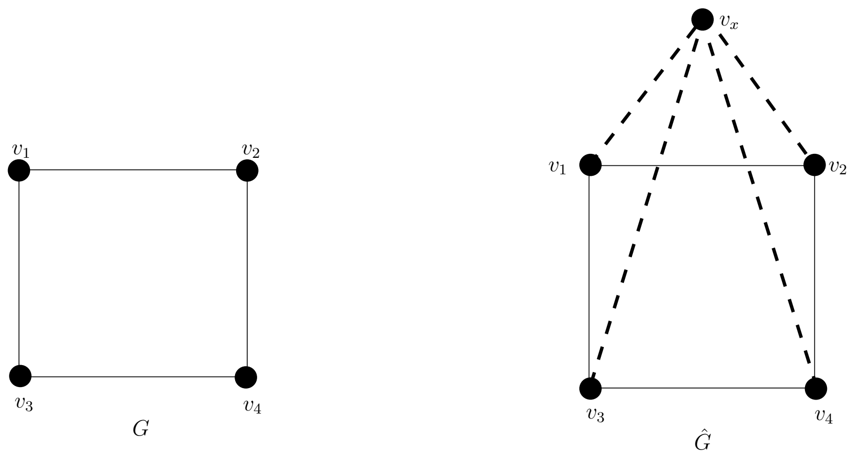


Figure 4.1: Graph G and the vertex-join $\hat{G}$.

4.3 Properties

In this section, we now state and prove some basic properties of the vertex-join of a graph.

It is clear that in the construction of the vertex-join of a graph, we create new cycles, in particular triangles. Thus, we can write the set of triangles of the vertex-join, $T(\hat{G})$, as $T(\hat{G}) = T(G) \cup T'(\hat{G})$, where $T(G)$ is the set of triangles of the graph G and $T'(\hat{G})$ is the set of triangles of $\hat{G}$ which are not in $T(G)$. Thus $T(\hat{G})$, can be partitioned into two sets, $T(G)$ and $T'(\hat{G})$.

Lemma 4.3.1. *Let G be a graph of order n with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{v_iv_j \mid$ for some $i, j \in \{1, \ldots, n\}\}$. Then each edge of G, v_iv_j, will be an edge of a new triangle in $\hat{G}$ which is not in G.*

Proof. Let $v_iv_j \in E(G)$. By Definition 4.2.1 of $\hat{G}$, then $v_iv_x, v_jv_x \in E(\hat{G})$. Since $E(G) \subseteq E(\hat{G})$, this implies $v_iv_j \in E(\hat{G})$. Hence we have that $\{v_iv_x, v_jv_x, v_iv_j\} \subseteq E(\hat{G})$ which is an edge set of a triangle by Definition 3.2.1. But this triangle is not in G since vertex $v_x \notin V(\hat{G})$. □

Proposition 4.3.2. *Let G be a graph of order n and size m. Let $T'(\hat{G})$ be the set of triangles in $\hat{G}$ but not in G. Then $|T'(\hat{G})| = m$.*

Proof. Let $\hat{G}$ be the vertex-join of the graph G, $V'(\hat{G}) = V(\hat{G}) - V(G)$ and $E'(\hat{G}) = E(\hat{G}) - E(G)$. Thus by Definition 4.2.1, $E'(\hat{G}) = \{v_1v_x, v_2v_x, \ldots, v_nv_x\}$ and $E(G) = \{v_iv_j \mid i, j \neq x \text{ and for some } i, j \in \{1, \ldots, n\}\ \}$. By Definition 3.2.1, a triangle is a three element edge set $\{v_iv_j, v_iv_k, v_jv_k\}$ on a three vertex set $\{v_i, v_j, v_k\}$. Hence there are three cases of elements of $T'(\hat{G})$:

Case 1. We first look at the case where we have three element edge set where two edges are in $E(G)$ and one edge in $E'(\hat{G})$. Say $v_iv_j, v_lv_k \in E(G)$ and $v_pv_x \in E'(\hat{G})$. If $v_iv_j = v_lv_k$, we have parallel edges so it is not a simple graph. So we consider $v_iv_j \neq v_lv_k$, however we already have 3 distinct vertices as endpoints, so adding v_p, v_x will add another vertex v_x as an endpoint and v_p does not matter in this case. This contradicts the definition of a triangle, since we have at least four element vertex set.

Case 2. Secondly we consider the case were we have three element edge set where all three edges are in $E'(\hat{G})$. Say $\{v_iv_x, v_jv_x, v_kv_x\}$ where $i \neq j \neq k$. This is a contradiction since we have a four element vertex set.

Case 3. Lastly we consider three element edge set where one edge is in $E(G)$ and two edges in $E'(\hat{G})$, say we have $\{v_iv_j, v_lv_x, v_kv_x\}$. If $\{i, j\} \neq \{l, k\}$, there is a contradiction, we have four vertices. But if $\{i, j\} = \{l, k\}$ we have $\{v_iv_j, v_iv_x, v_jv_x\}$ on three vertex set $\{v_i, v_j, v_x\}$ thus giving a triangle.

Hence since there are m edges of the form v_iv_j, the result follows that $|T'(\hat{G})| = m$. □

Corollary 4.3.3. *Let G be a graph of size m and $\hat{G}$ its vertex-join. Then each element of $T'(\hat{G})$ will have vertex v_x.*

Corollary 4.3.4. *Let G be a graph of size m and $\hat{G}$ the vertex-join of G. Let $T(G)$ be the set of triangles of G. Then $|T_{\Delta}(G)| = |T(G)| + m$.*

Proposition 4.3.5. *Let G be a graph of size m and $\hat{G}$ the vertex-join of G. Then the triangle graph of $\hat{G}$, $T_{\Delta}(\hat{G})$ has a subgraph isomorphic to a complete graph K_{m-1}*

Proof. Let $T'(\hat{G})$ be the set of triangles of $\hat{G}$ which are not in G. By Proposition 4.3.2, $|T'(\hat{G})| = m$. By Corollary 4.3.3, each element of $T'(\hat{G})$ has vertex v_x. Hence by construction of $T_{\Delta}(\hat{G})$, each vertex of $T_{\Delta}(\hat{G})$ which is in $T'(\hat{G})$ will be connected to the other $m-1$ vertices. □

4.4 Examples of Classes

In this section, we redefine some classes of graphs which are vertex-joins of other classes of graphs. In particular, we discuss the classes already discussed in Chapter 3. Thereafter, we verify our propositions in Chapter 3 by applying our results in Section 4.3.

We start by giving an alternative definition of a wheel graph as a vertex-join of another graph.

Definition 4.4.1. Let G be a cycle graph C_{n-1}. We define a wheel graph of order n, W_n to be the vertex-join of C_{n-1}.

We now verify one of the properties of Proposition 3.3.1 in Chapter 3 for wheel graphs. In Proposition 4.3.2, G is equal to C_{n-1} with $m = n-1$ and $\hat{G}$ is W_n. C_{n-1} does not have any triangles. Hence, W_n has $m = n-1$ more triangles than C_{n-1}. Verifying Proposition 3.3.1, that W_n has $n-1$ triangles.

Example 4.4.2. The graph in Figure 4.2 illustrates a cycle graph C_5 and the vertex-join $\hat{C}_5$. We note that $\hat{C}_5$ is equal to a wheel graph W_5.

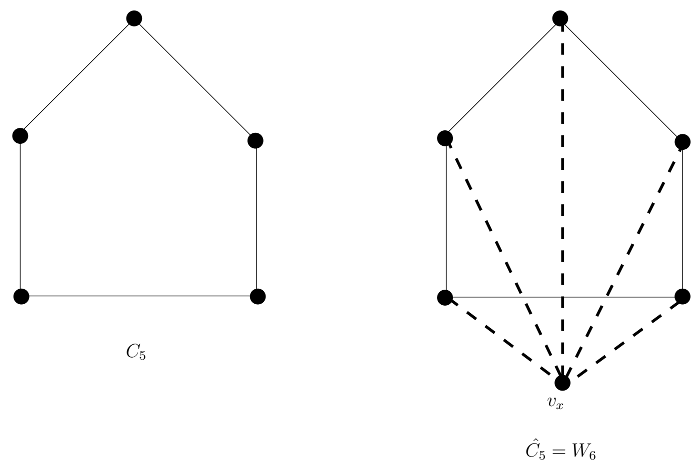


Figure 4.2: Graph C_5 and vertex-join $\hat{C}_5$.

We now give an alternative definition of a fan graph as a vertex-join of another graph.

Definition 4.4.3. Let G be a path graph P_{n-1}. We define a fan graph, f_n to be the vertex-join of P_{n-1}.

We now verify one of the properties of Proposition 3.3.18 in Chapter 3 for a fan graph. In Proposition 4.3.2, G is a fan and is equal to P_{n-1} with $m = n-1$ and $\hat{G}$. P_{n-1} does not have any triangles. Hence f_n has $m = n - 1$ more triangles than P_{n-1} verifying Proposition 3.3.18 that f_n has $n - 1$ triangles.

Example 4.4.4. The graph G given in Figure 4.3 is a path graph P_4 and the vertex-join of P_4.

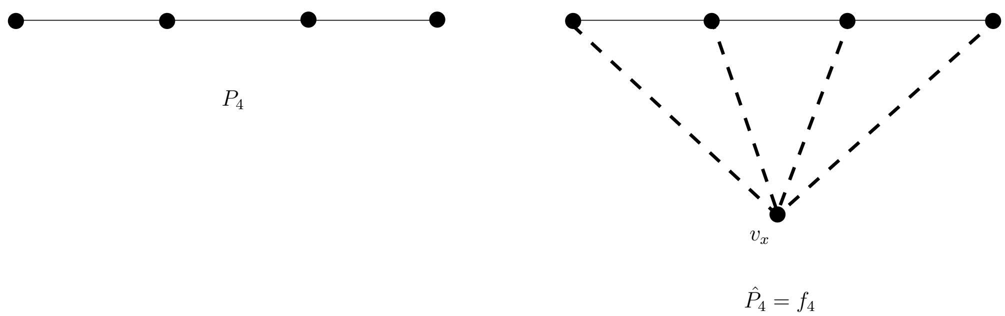


Figure 4.3: Graph P_4 and vertex-join $\hat{P}_4$.

4.5 Conclusion

In this chapter, we discussed the concept of the vertex-join of a graph. In Section 4.3, we stated and proved some properties of the triangle graph of a vertex-join. We concluded the chapter by giving alternative definitions of wheel graphs and fan graphs as vertex joins. We verified some of the properties stated in Chapter 3 for wheel graphs and fan graphs.

Chapter 5

Conclusion

In this dissertation, we set out to study derived graphs. We first looked at the historical background of graphs and derived graphs. Then we discussed some basic definitions and terms most commonly used in graph theory choosing the notations to be followed where a term has different notations in the literature . Thereafter, we looked at some classes of graphs which were useful in writing up this dissertation. We discussed derived graphs that come from different kinds of graph operations. We then discussed intersection graphs and chose only to discuss one type, interval graphs. We finally mentioned chordal graphs which some authors classify them under intersection graphs, but we chose to classify them under graph operations.

We noted that there are many graph operations but we chose to concentrate our study on line graphs in Chapter 2. With line graphs, we studied some known properties for certain classes of graphs mentioned in Chapter 1. Of much interest was the alternative ways of finding the number of edges of the line graph of a certain class of graph.

In Chapter 3, we discussed the main structure of this dissertation, the triangle graph of a graph. We established some notations which were used throughout from Chapter 3 to the end. Thereafter, we defined a graph operation called the triangle graph of a graph. We clarified the difference between the triangle graph of a graph and the cycle graph of a graph, where the cycle is of order 3. We then stated and proved a few properties of a triangle graph of a graph. Furthermore, we discussed triangle graphs of certain classes of graphs, namely; wheel graphs, helm graphs, sun flower graphs and friendship graphs.

Finally, in Chapter 4, we first discussed a well known graph operation called vertex-join of a graph G. Thereafter, we gave some general properties of a triangle graph of a vertex-join of a general graph. In addition, we gave alternative definitions of a wheel graph and a fan graph as vertex-joins of some classes of graphs. We concluded by verifying some properties from Chapter 3 for wheel graphs and fan graphs.

For further studies, one can address the following questions on triangle graph of a graph.

1. When can a triangle graph of G be isomorphic to G?
2. Can you determine classes of graphs such that the triangle graph of G is bigger in terms of size than the original graph G?
3. Is any graph G a triangle graph of some other graph?

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